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Overlaps of the biorthogonal $u_q(3)$ coupling coefficients and related basic hypergeometric and other q-factorial series

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Abstract. The intermediate expansion technique is developed for the overlap coefficients of different biorthogonal coupled states and coupling coefficients of the quantum algebra $u_q(3)$ and group SU(3) with repeating irreducible representations and the role of separate basic and classical hypergeometric functions is demonstrated. Some expansion coefficients such as new prime overlap functions (equivalent to the bilinear combinations of the *different* boundary isofactors) are expressed in terms of the balanced (Saalschützian) basic ${}_5\phi_4$ or classical ${}_5F_4(1)$ hypergeometric series. These new overlap functions, their analytical inversion, some known triangular expansion matrices, transition matrices related to definite $u_q(2)$ Racah coefficients, the well-poised ${}_{10}\phi_9$ or ${}_9F_8(1)$ series and compositions of q-factorial series resembling the well-poised ${}_9\phi_8$ and ${}_{11}\phi_{10}$ series or equivalent to ${}_8F_7(-1)$ and ${}_{10}F_9(-1)$ series provide themselves as generators for mutual expansion of different non-orthonormal systems of $u_q(3)$ and SU(3) isofactors, which may be orthonormalized to the paracanonical or (with a definite caution) canonical version.

1. Introduction

The relationships of the coupling (Clebsch–Gordan–Wigner) and recoupling (Racah) coefficients of the quantum group $u_q(2)$ with the basic hypergeometric functions ${}_3\phi_2(q)$ and ${}_4\phi_3(q)$, respectively, are well known (see e.g. Groza *et al* [1], Kachurik and Klimyk [2], Rajeswari and Srinivasa Rao [3]), as well as the relations of the Clebsch–Gordan and Racah coefficients of the group SU(2) with the classical hypergeometric functions ${}_3F_2(1)$ and ${}_4F_3(1)$. In recent years, the coupling coefficients of the quantum groups $u_q(n)$ and, particularly, $u_q(3)$ were considered by several authors (Smirnov *et al* [4], Gould and Biedenharn [5], Smirnov and Kharitonov [6, 7] and expressed as multiple sums (Committeeišauskas and Smirnov [8], Committeeišauskas [9, 10]), some of which are separately related to the basic hypergeometric series. In contrast with the multiplicity-free $u_q(n)$ cases [8] (including the general $u_q(2)$ case), the orthogonalization and normalization problem of the coupling coefficients and the coupled states of $u_q(3)$ is non-trivial, when the repeating irreducible representations (irreps) appear in the coproduct decomposition, similarly as for the compact Lie group SU(3). The analytical biorthogonal systems of the $u_q(3)$ isoscalar factors (isofactors) emerge quite naturally [9]. However, the overlap coefficients (which form the metric tensor for the dual bases or coupled states and may also be also correlated with the K-matrix technique of the vector coherent state theory—cf Deenen and Quesne [11], Quesne [12], Le Blanc and Rowe [13], Rowe *et al* [14]) are necessary for the orthonormalization (by means of the Gram–Schmidt procedure or using the square root of the overlap matrix

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as the K-matrix, respectively), unless the explicit orthogonalization coefficients (Ališauskas [15, 16]) of the SU(3) isofactors, (with the partition dependent generalization of the balanced hypergeometric functions ${}_4F_3(1)$—the $A_\lambda\binom{a,b,d,e}{c}$-functions of Louck and Biedenharn [17]—see Louck *et al* [18], Biedenharn *et al* [19] as the numerator–denominator polynomials) will be generalized to the $\mathrm{u}_q(3)$ isofactors. The expressions of the overlaps related to the bilinear combinations of special recoupling coefficients [9] and derived by means of the projection operators of the complementary quantum algebras (Quesne [20], Smirnov and Tolstoy [21]), include the double sums, with separate sums related to rather complicated (well-poised) basic hypergeometric series, but without any definite sense in the representation theory of $\mathrm{u}_q(3)$.

Otherwise, the overlaps of the biorthogonal states for the two-parametric irreps of $\mathrm{SU}(n) \supset \mathrm{SO}(n)$ and $\mathrm{Sp}(4) \supset \mathrm{U}(2)$ were expressed using the intermediate expansion (in terms of the auxiliary states—see [22–25]) technique. By analogy with the SU(3) case (Ališauskas [26]), the triangular expansion matrices of definite non-orthogonal $\mathrm{u}_q(3)$ isofactors (which correspond to the same version of the orthogonal—paracanonical isofactors) are formed by overlaps of certain (dual) coupled states—the boundary (extreme) values of definite non-orthogonal isofactors [9]. However, they are insufficient for expansion of the non-orthogonal $\mathrm{u}_q(3)$ isofactors in terms of the mutually dual ones. Another problem is presented by the explicit orthonormalization of the SU(3) and $\mathrm{u}_q(3)$ isofactors, constructed by the method of Draayer and Akiyama [27], since conjecture [28] about the possibility of the immediate construction of orthogonal SU(3) canonical tensor operators for non-extreme values of the multiplicity label t was not confirmed (see [10]).

We will consider some elementary expansion coefficients and overlaps of the different non-orthogonal sets of $\mathrm{u}_q(3)$ and SU(3) isofactors (which lead to the sets of the paracanonical or canonical orthonormal $\mathrm{u}_q(3)$ and SU(3) isofactors, respectively) and express them in terms of basic (or classical) hypergeometric series or related q-factorial series. The role of the different rearrangement and summation formulae of q-factorial series is also demonstrated. Section 2 is devoted to the main definitions and recursive construction of the biorthogonal systems of isofactors, specified to the $\mathrm{u}_q(3)$ case.

In section 3 we derive a new interesting expression in terms of the balanced (Saalschützian) ${}_5\phi_4(q)$ basic hypergeometric series (cf Gasper and Rahman [29]) for overlaps of different non-orthogonal systems of $\mathrm{u}_q(3)$ isofactors (but yielding the same version of the paracanonical isofactors), which are equivalent to the bilinear combinations of extreme (boundary) $\mathrm{u}_q(3)$ isofactors of different types. This expression may be used as the last important but indispensable link for overlaps of the definite (the same) non-orthogonal system of $\mathrm{u}_q(3)$ isofactors, i.e. for the bilinear combination of extreme $\mathrm{u}_q(3)$ isofactors of the same type in terms of the double sum, with the second sum equivalent to the well-poised ${}_{10}\phi_9(q)$ basic hypergeometric series or to the well-poised ${}_9F_8(1)$ classical hypergeometric series in the SU(3) case. Moreover, the new expansion matrix may be inverted by means of the analytical inversion (Ališauskas [26, 30]) substitutions. Hence, an interesting biorthogonality relation for the terminating balanced ${}_5\phi_4(q)$ basic hypergeometric series is also derived.

In section 4, some overlaps—the expansion coefficients between the different families of the biorthogonal systems of $\mathrm{u}_q(3)$ isofactors (which lead to the different versions of paracanonical or canonical isofactors)—are presented in terms of the $\mathrm{u}_q(2)$ recoupling (Racah or q $6j$-) coefficients, including those which lead to the isofactors—reduced matrix elements of the $\mathrm{u}_q(3)$ canonical tensor operators. In particular, some $\mathrm{u}_q(3)$ canonical isoscalar factors with the minimal null space of the $\mathrm{u}_q(3)$ canonical tensor operators are derived, including normalized reduced matrix elements of extreme components of the self-adjoint unit canonical tensor operators.

Section 5 is devoted to the expansion coefficients of the (orthonormal) $\mathrm{u}_q(3)$ canonical isofactors with the maximal null space (as defined in [10]) and expansion of the general (non-orthonormal) construction of isofactors (Plkshauskas [10, 28]) which leads to the $\mathrm{u}_q(3)$ canonical tensor operators. The corresponding overlaps are expressed in terms of the q-factorial series, resembling the well-poised ${}_9\phi_8$ (which, in turn, have been rearranged from the double sum, each separately of ${}_3\phi_2$ type) and ${}_{11}\phi_{10}$ basic hypergeometric series.

2. Defining relations and summary of some previous results

As in [8–10], we use the Cartan–Weyl generators $E_{ik}(i, j, k = 1, 2, 3)$ of the unitary quantum algebra $\mathrm{u}_q(3) = U_q(u(3))$, which satisfy the commutation relations

$$[E_{ii}, E_{kk}] = 0 \qquad [E_{ii}, E_{jk}] = \delta_{ij} E_{ik} - \delta_{ik} E_{ji} \tag{2.1a}$$

$$[E_{ik}, E_{ki}] = [E_{ii} - E_{kk}]. \tag{2.1b}$$

Here and below $[x]$ and $[x]!$ are, respectively, the q-numbers and q-factorials:

$$[x] = \frac{(q^x - q^{-x})}{(q - q^{-1})} \qquad [x]! = [x][x-1]\ldots[2][1] \qquad [1]! = [0]! = 1 \tag{2.2a}$$

$$(\alpha|q)_n = \prod_{k=0}^{n-1}[\alpha + k] = [\alpha][\alpha+1]\ldots[\alpha+n-1] = \frac{[\alpha+n-1]!}{[\alpha-1]!} \tag{2.2b}$$

which are symmetric under substitution $q \leftrightarrow q^{-1}$.

The composite generators may be expressed in terms of the q-deformed commutators

$$E_{13} = [E_{12}, E_{23}]_q = E_{12}E_{23} - qE_{23}E_{12} \tag{2.3a}$$

$$E_{31} = [E_{32}, E_{21}]_{q^{-1}} = E_{32}E_{21} - q^{-1}E_{21}E_{32} \tag{2.3b}$$

and satisfy the Serre identities. Generators E_{12} and E_{21} are chosen for the canonical $\mathrm{u}_q(2)$ subalgebra, used for labelling of the basis states in the U basis.

We use the coproduct expansion rules

$$\Delta(E_{ii}) = E_{ii} \otimes 1 + 1 \otimes E_{ii} \tag{2.4a}$$

$$\Delta(E_{ii+1}) = E_{ii+1} \otimes q^{1/2(E_{ii} - E_{i+1,i+1})} + q^{-1/2(E_{ii} - E_{i+1,i+1})} \otimes E_{ii+1} \tag{2.4b}$$

$$\Delta(E_{i+1i}) = E_{i+1i} \otimes q^{1/2(E_{ii} - E_{i+1,i+1})} + q^{-1/2(E_{ii} - E_{i+1,i+1})} \otimes E_{i+1i} \tag{2.4c}$$

as well as special coproduct formulae (Smirnov *et al* [4]) for $\Delta(E_{13})$ and $\Delta(E_{31})$.

We use here the same notations for irreps and basis states of $\mathrm{u}_q(3)$ as were used in [10, 26, 28] with $(a\ b)$ for the mixed tensor irreps

$$a = m_{13} - m_{23} \qquad b = m_{23} - m_{33} \qquad \text{where } [m_{13}, m_{23}, m_{33}] \tag{2.5a}$$

is a Young frame (partition). The basis states are labelled by the hypercharge y, the isospin i and its projection i_z:

$$y = m_{12} + m_{22} - \tfrac{2}{3}(m_{13} + m_{23} + m_{33}) \qquad i = \tfrac{1}{2}(m_{12} - m_{22})$$
$$i_z = m_{11} - \tfrac{1}{2}(m_{12} + m_{22}) \tag{2.5b}$$

where m_{ij} are the Gelfand–Tsetlin parameters (see [5]). The parameter

$$z = \tfrac{1}{3}(b - a) - \tfrac{1}{2}y = m_{23} - \tfrac{1}{2}(m_{12} + m_{22}) \tag{2.5c}$$

is sometimes more convenient than y, because

$$i \pm z \geqslant 0 \qquad a + z - i \geqslant 0 \qquad b - z - i \geqslant 0 \tag{2.5d}$$

are integers. For the state of irrep $(a\ b)$ in the coproduct $(a'b')\otimes(a''b'')$ decomposition,

$$z = z' + z'' + v \qquad \text{where } v = \tfrac{1}{3}(a' - b' + a'' - b'' - a + b). \tag{2.6}$$

The parameters of the highest weight state (HWS) accept the values

$$y_0 = \tfrac{1}{3}(a + 2b) \qquad i_0 = \tfrac{1}{2}a = -z_0 \tag{2.7a}$$

while for the lowest weight state (LWS)

$$\bar{y}_0 = -\tfrac{1}{3}(2a + b) \qquad \bar{i}_0 = \tfrac{1}{2}b = \bar{z}_0 \tag{2.7b}$$

and for the maximal isospin state (MIS)

$$y_\mathrm{m} = \tfrac{1}{3}(a - b) \qquad i_\mathrm{m} = \tfrac{1}{2}(a + b) \qquad z_\mathrm{m} = \tfrac{1}{2}(b - a). \tag{2.7c}$$

The multiplicity of irrep $(a\ b)$ in the coproduct $(a'b')\otimes(a''b'')$ decomposition is equal to

$$r = \min r_{\alpha\beta\gamma} + 1 \qquad (\alpha = 1, 2, 3;\ \beta = 1, 2, 3;\ \gamma = 1, 2) \tag{2.8a}$$

where $r_{\alpha\beta\gamma}$ form the following $3\times 3\times 2$ array (with the third dimension represented by a skew shift in plane):

$$|r_{\alpha\beta\gamma}| = \left|\begin{array}{cccccc} r_{111} & & b-v & & b' & \\ & r_{112} & & b & & b'+v \\ a & & r_{221} & & a''-v & \\ & a+v & & r_{222} & & a'' \\ a'-v & & b'' & & r_{331} & \\ & a' & & b''+v & & r_{332} \end{array}\right| \tag{2.8b}$$

$$r_{111} = b' - a'' + a + v \qquad r_{112} = a' - b'' + b - v \qquad r_{221} = a - a' + b'' + v$$
$$r_{222} = a'' - b' + b - v \qquad r_{331} = b' + b'' - b + v \qquad r_{332} = a' + a'' - a - v$$

with equidistant parameters in the layers, rows and columns:

$$r_{\alpha\beta 2} - r_{\alpha\beta 1} = r_{\alpha'\beta' 2} - r_{\alpha'\beta' 1} = v \qquad r_{\alpha\beta\gamma} - r_{\alpha\beta'\gamma} = r_{\alpha'\beta\gamma} - r_{\alpha'\beta'\gamma}. \tag{2.8c}$$

Now let us present some expressions for the biorthogonal isofactors of $\mathrm{u}_q(3)$ derived by means of the recoupling technique [9]†. Equations (3.1) and (3.2b) present the non-orthogonal isofactors (which are equivalent to the bilinear combination of orthogonal isofactors) in terms of the multiplicity-free isofactors of $\mathrm{u}_q(3)$ [8]‡ and recoupling coefficients of $\mathrm{u}_q(2)$

$$\begin{aligned}&\begin{bmatrix} (a'b') & (a''b'') & {}_{-,+,\tilde{J}}(ab) \\ y'i' & y''i'' & yi \end{bmatrix}_q^{(3)} \\ &\qquad \equiv \sum_\rho \begin{bmatrix} (a'b') & (a''b'') & {}^{\rho}(ab) \\ (-i'_0)i'_0 & (\bar{i}''_0)\bar{i}''_0 & (\tilde{z})\tilde{J} \end{bmatrix}_q^{(3)} \begin{bmatrix} (a'b') & (a''b'') & {}_{\rho}(ab) \\ (z')i' & (z'')i'' & (z)i \end{bmatrix}_q^{(3)} \\ &\qquad = q^{Q_1(a'b'a''b''ab;\tilde{J}\tilde{z})}\left(\frac{[a''+b''+1]!}{[a''+1][2\tilde{J}+1]}\right.\end{aligned} \tag{2.9a}$$

† Note that the second λ should be corrected into Λ in the right-hand side of (3.2a) of [9], $z_0 = z' + z'' + v$ and $\tilde{z}' = \frac{1}{2}(a'' - a) - v$ in (4.7), $\tilde{I}'$ should be replaced by $\tilde{I}$ in (4.15), the last λ in the denominator under the sum sign on the right-hand side of (5.3) should be replaced by μ and second $S_{n,n}[\lambda'; \Lambda]$ in the right-hand side of (5.56) should be corrected into $S_{n,n}[\lambda; \Lambda]$.

‡ Note that the last μ in the denominator of the right-hand side of (3.14) of [8] should be changed to μ'; the numerator square root factor $[\lambda'_k]!$ is omitted on the right-hand side of (3.20), as well as the product of the numerator square root factors $[\lambda'_1 - \lambda'_j - 1 + j]!$ $(2 \leqslant j \leqslant k)$ on the right-hand side of (3.21); factor $[2i+1]$ is omitted in the numerator under the square root of (4.5) and in the denominator under the square root of (4.8).

$$\times \frac{[a+1][b+1][a+b+2][\bar{b}]![\bar{b}+2\tilde{J}+1]!}{[b'+b''-b+v]![b'+b''+v+1]![b'+b''+a+v+2]!}\Bigg)^{1/2}$$

$$\times \sum_{j_2,j_3,J,Z} \begin{bmatrix} (a'b') & (b'',0) & (2\tilde{J},\bar{b}) \\ (z')i' & (-j_2)j_2 & (Z)J \end{bmatrix}_q^{(3)} \begin{bmatrix} (2\tilde{J},\bar{b}) & (a''+b'',0) & (ab) \\ (Z)J & (-j_3)j_3 & (z)i \end{bmatrix}_q^{(3)}$$

$$\times \begin{bmatrix} (b'',0) & (a''+b'',0) & (a''b'') \\ (-j_2)j_2 & (-j_3)j_3 & (z'')i'' \end{bmatrix}_q^{(3)} U_2(i'j_2ij_3;\, Ji'')_q \qquad (2.9b)$$

with two independent summation parameters in the right-hand side, where

$$j_2+j_3=b''-z'' \qquad Z=z'-j_2+\bar{b}-b' \qquad \bar{b}=\tfrac{1}{2}(a'+b'')+b'-\tilde{J}$$

$$\tilde{z}=\tfrac{1}{2}(b''-a')+v \qquad a+\tilde{z}-\tilde{J}\geqslant 0 \qquad b-\tilde{z}-\tilde{J}\geqslant 0$$

$$\tfrac{1}{2}(a'+b'')-\tilde{J}\geqslant 0 \qquad \tilde{J}\pm\tfrac{1}{2}(a'-b'')\geqslant 0 \qquad \tilde{J}\pm\tilde{z}\geqslant 0$$

$$Q_1(a'b'a''b''ab;\,\tilde{J}\tilde{z})=\tfrac{1}{2}\{\tilde{J}(\tilde{J}+1)+\tilde{z}(3\tilde{z}+2a-2b)-ab+\tfrac{1}{2}(a'+b'')$$
$$+a''+b'-a-b\}. \qquad (2.10)$$

The $-$ and $+$ signs and their positions in the multiplicity label $-,+,\tilde{J}$ (presented on the left-hand side of (2.9) as the subscript $-,+,\tilde{J}$) indicate the signs and positions of extreme z' and z'' in (2.9a), where ρ is an arbitrary (orthogonal or biorthogonal) multiplicity label.

Expressions for the non-orthogonal isofactors of the dual type (which satisfy the definite boundary conditions—see section 4 of [9]) may be presented as follows:

$$\begin{bmatrix} (a'b') & (a''b'') & {}^{\rho}(ab) \\ y'i' & y''i'' & yi \end{bmatrix}_q^{(3)} = N^{\rho}[a'b'a''b'';ab]$$

$$\times \sum_{j_2,j_3,J,Z} \begin{bmatrix} (a'b') & (0b'') & (\tilde{a}\tilde{b}) \\ (z')i' & (j_2)j_2 & (Z)J \end{bmatrix}_q^{(3)} \begin{bmatrix} (\tilde{a}\tilde{b}) & (a''0) & (ab) \\ (Z)J & (-j_3)j_3 & (z)i \end{bmatrix}_q^{(3)}$$

$$\times \begin{bmatrix} (0b'') & (a''0) & (a''b'') \\ (j_2)j_2 & (-j_3)j_3 & (z'')i'' \end{bmatrix}_q^{(3)} U_2(i'j_2ij_3;\, Ji'')_q \qquad (2.11)$$

where $j_2-j_3=z''$. We take in (2.11) for $a'+2b'+a''-b''-a-2b\geqslant 0$ the intermediate irreps

$$(\tilde{a},\tilde{b})=(2\tilde{J},\tfrac{1}{2}(a'-b'')+b'-\tilde{J}) \qquad (2.12a)$$

(correlated with the multiplicity label (superscript) $\rho=-,+,\tilde{J}$) and renormalization factor

$$N^{-,+,\tilde{J}}[a'b'a''b'';ab]=\left(\begin{bmatrix} (a'b') & (0b'') & (\tilde{a}\tilde{b}) \\ (-i_0')i_0' & (\bar{i}_0'')\bar{i}_0'' & (-\tilde{J})\tilde{J} \end{bmatrix}_q^{(3)} \begin{bmatrix} (\tilde{a}\tilde{b}) & (a''0) & (ab) \\ (-\tilde{J})\tilde{J} & (0)0 & (\tilde{z})\tilde{J} \end{bmatrix}_q^{(3)}\right)^{-1} \qquad (2.12b)$$

when for $a'+2b'+a''-b''-a-2b\leqslant 0$ we may use the intermediate irreps

$$(\tilde{a},\tilde{b})=(2\tilde{I}',\tfrac{1}{2}(a-a'')+b-\tilde{I}') \qquad (2.13a)$$

the multiplicity label (superscript) $\rho=\tilde{I}',-,-$ and renormalization factor

$$N^{\tilde{I}',-,-}[a'b'a''b'';ab]=\left(\begin{bmatrix} (a'b') & (0b'') & (\tilde{a}\tilde{b}) \\ (\tilde{z}')\tilde{I}' & (0)0 & (-\tilde{I}')\tilde{I}' \end{bmatrix}_q^{(3)}\right)^{-1} \qquad (2.13b)$$

where $\tilde{z}'=\frac{1}{2}(a''-a)-v$. Equation (2.11) with inserted (2.12a) and (2.12b) turns into $\delta_{i,\tilde{J}}$ for $i'=-z'=\frac{1}{2}a'$, $i''=z''=\frac{1}{2}b''$, but expression (2.11) together with (2.12a) and (2.12b) gives $\delta_{i',\tilde{I}'}$ for $i''=-z''=\frac{1}{2}a''$, $i=-z=\frac{1}{2}a$. The multiplicity free isofactors on the right-hand side of (2.9b) and (2.11) include double, single and no sums, respectively, but the corresponding isofactors on the right-hand side of (2.12b) and (2.13b) may be expressed without any sum, using their symmetry properties [8, 10].

3. Elementary overlaps of biorthogonal coupled states and basic hypergeometric series

We present the triangle direct and inverse overlap matrices (in accordance with comments and q-phase (4.13) at the end of section 4 of [9] and equation (2.11) of [26]) as special cases of (2.11), with inserted (2.12) or (2.13):

$$
\begin{aligned}
(\eta^{\tilde{I}',-,-}|\eta_{-,+,\tilde{J}})_q &\equiv \begin{bmatrix} (a'b') & (a''b'') & {}^{\tilde{I}',-,-}(ab) \\ y_0'i_0' & \overline{y}_0''\overline{i}_0'' & \tilde{y}\tilde{J} \end{bmatrix}_q^{(3)} \\
&= (-1)^{\tilde{I}'+\tilde{z}'} q^{a''/2+b''} B(a'b'a''b''ab;\, \tilde{I}'\tilde{z}',\, \tilde{J}\tilde{z})([2\tilde{I}'+1][a+1])^{1/2} \\
&\quad\times \frac{[\tilde{J}+\tilde{I}'+\frac{1}{2}(b'-b+v)]!(b'-b+v|q)_{(b-b'-v)/2-\tilde{J}+\tilde{I}'}}{[\frac{1}{2}(b-b'-v)-\tilde{J}+\tilde{I}']![\frac{1}{2}(b-b'-v)+\tilde{J}+\tilde{I}'+1]!}
\end{aligned} \qquad (3.1a)
$$

$$
\begin{aligned}
(\eta_{\tilde{J}',-,-}|\eta^{-,+,\tilde{I}})_q &\equiv \begin{bmatrix} (a'b') & (a''b'') & {}^{-,+,\tilde{I}}(ab) \\ \tilde{y}'\tilde{J}' & y_0''i_0'' & y_0 i_0 \end{bmatrix}_q^{(3)} \\
&= (-1)^{a''-a+(a'-b'')/2+\tilde{I}} q^{-a''/2-b''} B^{-1}(a'b'a''b''ab;\, \tilde{J}'\tilde{z}',\, \tilde{I}\tilde{z})([2\tilde{J}'+1])^{1/2} \\
&\quad\times \frac{[2\tilde{I}+1][\tilde{I}+\tilde{J}'+\frac{1}{2}(b-b'-v)]!(b-b'-v|q)_{(b'-b+v)/2-\tilde{J}'+\tilde{I}}}{([a+1])^{1/2}[\frac{1}{2}(b'-b+v)+\tilde{I}-\tilde{J}']![\frac{1}{2}(b'-b+v)+\tilde{I}+\tilde{J}'+1]!}
\end{aligned} \qquad (3.1b)
$$

where

$$
\begin{aligned}
B(a'b'a''b''ab;\, \tilde{I}'\tilde{z}',\, \tilde{J}\tilde{z}) &= B^{-1}(abb''a''a'b';\, \tilde{J}\tilde{z},\, \tilde{I}'\tilde{z}') = q^{Q_1(a'b'a''b''ab;\tilde{J}\tilde{z})-Q_1(abb''a''a'b';\tilde{I}'\tilde{z}')} \\
&\quad\times \frac{\nabla[\frac{1}{2}b'',\, \frac{1}{2}a',\, \tilde{J}]}{\nabla[\frac{1}{2}a'',\, \frac{1}{2}a,\, \tilde{I}']} \frac{H[ab\tilde{J}\tilde{z}]}{H[a'b'\tilde{I}'\tilde{z}']} \left(\frac{[b']![a'+b'+1]![a'']![\tilde{I}'-\tilde{z}']![\tilde{J}+\tilde{z}]!}{[b'']![b]![a+b+1]![\tilde{I}'+\tilde{z}']![\tilde{J}-\tilde{z}]!} \right)^{1/2}
\end{aligned} \qquad (3.2)
$$

$$\tilde{z}' = \tfrac{1}{2}(a''-a) - v, \qquad \tilde{z} = \tfrac{1}{2}(b''-a') + v,$$

$$\nabla[abc] = \left(\frac{[a+b-c]![a-b+c]![a+b+c+1]!}{[b+c-a]!} \right)^{1/2} \qquad (3.3)$$

$$H[abiz] = ([a+z-i]![a+z+i+1]![b-z-i]![b-z+i+1]!)^{1/2} \qquad (3.4)$$

and the q-phase is expressed in terms of (2.10). We see that for $b'-b+v=0$ in (3.1a) $\tilde{J}=\tilde{I}'$ and in (3.1b) $\tilde{J}'=\tilde{I}$. In general, we may verify the biorthogonality relations

$$\sum_{\tilde{J}} (\eta^{\tilde{I}',-,-}|\eta_{-,+,\tilde{J}})_q (\eta^{-,+,\tilde{J}}|\eta_{\tilde{J}',-,-})_q = \delta_{\tilde{I}',\tilde{J}'} \qquad (3.5a)$$

$$\sum_{\tilde{J}'} (\eta^{-,+,\tilde{I}}|\eta_{\tilde{J}',-,-})_q (\eta^{\tilde{J}',-,-}|\eta_{-,+,\tilde{J}})_q = \delta_{\tilde{I},\tilde{J}} \qquad (3.5b)$$

for (3.1*a*) and (3.1*b*) straightforwardly, unless $\tilde{I}' > \tilde{J}'$ in (3.5*a*) or $\tilde{I} > \tilde{J}$ in (3.5*b*) when we should use the summation formula

$$\sum_j \frac{(-1)^{p_4-j}[2j+1][j-p_1-1]![j-p_2-1]![j-p_3-1]!}{[p_1+j+1]![p_2+j+1]![p_3+j+1]![p_4-j]![p_4+j+1]!} = \frac{[-p_1-p_2-2]![-p_2-p_3-2]![-p_1-p_3-2]!}{[p_1+p_4+1]![p_2+p_4+1]![p_3+p_4+1]![-p_1-p_2-p_3-p_4-3]!} \tag{3.6}$$

(cf special very well-poised basic hypergeometric series ${}_6\phi_5$ as (2.4.2) of [29]).

Now let us consider the overlaps of another kind. The first isofactor on the right-hand side of (2.9*b*) is proportional to

$$\left\{ \begin{matrix} \frac{1}{2}a'+\tilde{z}' & \tilde{I}' & \frac{1}{2}a' \\ \frac{1}{2}b'' & \tilde{J} & J \end{matrix} \right\}$$

for the fixed HWS of the second and resulting irreps on the left-hand side, with the summation parameters accepting values $j_2 = \frac{1}{2}b''$ and $j_3 = \frac{1}{2}(a''+b'')$. Then the remaining isofactors may be expressed without sum and (3.6) leads to the following expression for the overlaps:

$$\begin{aligned}
(\eta_{\tilde{I}',-,-}|\eta_{-,+,\tilde{J}})_q &\equiv \sum_\rho \begin{bmatrix} (a'b') & (a''b'') & {}^{\rho}(ab) \\ \tilde{y}'\tilde{I}' & y_0''i_0'' & y_0 i_0 \end{bmatrix}_q^{(3)} \begin{bmatrix} (a'b') & (a''b'') & {}_{\rho}(ab) \\ y_0'i_0' & \bar{y}_0''\bar{i}_0'' & \tilde{y}\tilde{J} \end{bmatrix}_q^{(3)} \\
&= (-1)^{a-a''+(a'-b'')/2+\tilde{J}} q^{Q_1(a'b'a''b''ab;\tilde{J}\tilde{z})+Q_1(abb''a''a'b';\tilde{I}'\tilde{z}')-a''/2-b''} \\
&\quad\times \frac{[a''+b''+1]!([a+1][b+1][a+b+2])^{1/2}}{\nabla[\frac{1}{2}b'',\frac{1}{2}a',\tilde{J}]\nabla[\frac{1}{2}a'',\frac{1}{2}a,\tilde{I}']H[a'b'\tilde{I}'\tilde{z}']H[ab\tilde{J}\tilde{z}]} \left(\frac{[\tilde{I}'-\tilde{z}']![\tilde{J}-\tilde{z}]!}{[\tilde{I}'+\tilde{z}']![\tilde{J}+\tilde{z}]!}\right)^{1/2} \\
&\quad\times([2\tilde{I}'+1][b']![a'+b'+1]![a'']![b'']![b+1]![a+b+2]!)^{1/2} \\
&\quad\times \sum_s \frac{[\tilde{J}+\tilde{z}+s]![\frac{1}{2}(a''-a)+\tilde{I}'+s]![a-s]![a'-v-s]!}{[s]![\tilde{J}-\tilde{z}-s]![\frac{1}{2}(a-a'')+\tilde{I}'-s]![v+s]!} \\
&\quad\times \frac{[b'-a''+a+v-s]!}{[b'+b''+a+v-s+2]!}.
\end{aligned} \tag{3.7}$$

Now the intermediate expansion technique leads to the expression containing a double sum for the overlaps of the coupled states of the same type

$$\begin{aligned}
(\eta_{-,+,\tilde{I}}|\eta_{-,+,\tilde{J}})_q &\equiv \sum_\rho \begin{bmatrix} (a'b') & (a''b'') & {}_{\rho}(ab) \\ y_0'i_0' & \bar{y}_0''\bar{i}_0'' & \tilde{y}\tilde{I} \end{bmatrix}_q^{(3)} \begin{bmatrix} (a'b') & (a''b'') & {}^{\rho}(ab) \\ y_0'i_0' & \bar{y}_0''\bar{i}_0'' & \tilde{y}\tilde{J} \end{bmatrix}_q^{(3)} \\
&= \sum_{\tilde{I}'} (\eta_{-,+,\tilde{I}}|\eta^{\tilde{I}',-,-})_q (\eta_{\tilde{I}',-,-}|\eta_{-,+,\tilde{J}})_q \\
&= (-1)^{\tilde{I}-\tilde{J}} q^{Q_1(a'b'a''b''ab;\tilde{I}\tilde{z})+Q_1(a'b'a''b''ab;\tilde{J}\tilde{z})} \\
&\quad\times[a+1][b+1][a+b+2][b']![a'+b'+1]![a'']![a''+b''+1]! \\
&\quad\times \frac{\nabla[\frac{1}{2}b'',\frac{1}{2}a',\tilde{I}]H[ab\tilde{I}\tilde{z}]}{\nabla[\frac{1}{2}b'',\frac{1}{2}a',\tilde{J}]H[ab\tilde{J}\tilde{z}]} \left(\frac{[\tilde{I}+\tilde{z}]![\tilde{J}-\tilde{z}]!}{[\tilde{I}-\tilde{z}]![\tilde{J}+\tilde{z}]!}\right)^{1/2} \\
&\quad\times \sum_{s,\tilde{i}'} \frac{[\tilde{J}+\tilde{z}+s]![\frac{1}{2}(a''-a)+\tilde{i}'+s]![a-s]![a'-v-s]!}{[s]![\tilde{J}-\tilde{z}-s]![\frac{1}{2}(a-a'')+\tilde{i}'-s]![v+s]!}
\end{aligned}$$

$$\times \frac{[b'-a''+a+v-s]![2\tilde{i}'+1][\tilde{i}'-\tilde{z}']!(-1)^{(b-b'-v)/2-\tilde{I}+\tilde{i}'}}{[b'+b''+a+v-s+2]![\tilde{i}'+\tilde{z}']!\nabla^2[\frac{1}{2}a'',\frac{1}{2}a,\tilde{i}']H^2[a'b'\tilde{i}'\tilde{z}']}$$
$$\times \frac{[\tilde{I}+\tilde{i}'+\frac{1}{2}(b'-b+v)]!(b'-b+v|q)_{(b-b'-v)/2-\tilde{I}+\tilde{i}'}}{[\frac{1}{2}(b-b'-v)-\tilde{I}+\tilde{i}']![\frac{1}{2}(b-b'-v)+\tilde{I}+\tilde{i}'+1]!}. \tag{3.8}$$

For $b-b'-v=0$ we have in (3.8) a single sum, since $\frac{1}{2}(b-b'-v)-\tilde{I}+\tilde{i}'=0$, and for $b-b'-v\geqslant 0$ all the terms on the right-hand side of (3.8) have the same sign (but the number of values accepted by $\tilde{I}$ and $\tilde{J}$ may exceed the multiplicity of irrep $(a\ b)$ in the coproduct $(a'b')\otimes(a''b'')$ decomposition).

Returning to equation (3.7), note that its SU(3) version may also be derived integrating over the group the product of three special SU(3) D-functions, presented by Norvaišas [31] in Holland's [32] (see also Pluhař *et al* [33]) parametrization. We see that the sum on the right-hand side of (3.7) corresponds to the balanced (Saalschützian) ${}_5\phi_4$ basic (or ${}_5F_4(1)$ classical) hypergeometric series, depending on eight free parameters. For $v\geqslant 0$ it may be written in terms of series

$$\frac{[A_1+n-1]![A_2+m-1]![N_1]![N_2]![N_3]!}{[n]![m]![A_3]![B-1]!}\,{}_5F_4\left[\begin{matrix}-n,-m,A_1+n,A_2+m,-A_3\\ -N_1,-N_2,-N_3,B\end{matrix};q,1\right] \tag{3.9a}$$

with

$$n=\tilde{J}-\tilde{z}\qquad m=\tfrac{1}{2}(a-a'')+\tilde{I}'\qquad A_1=2\tilde{z}+1\qquad A_2=a''-a+1$$
$$A_3=b'+b''+a+v+2\qquad B=v+1\qquad N_1=a$$
$$N_2=a'-v\qquad N_3=b'-a''+a+v. \tag{3.9b}$$

(For $v<0$, the summation parameter $s'=v+s$ should be used instead of s). We use here the series (cf Álvarez–Nodarse and Smirnov [34])

$${}_{p+1}F_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1}\\ \beta_1,\ldots,\beta_p\end{matrix};q,x\right]=\sum_k^{\infty}\frac{(\alpha_1|q)_k(\alpha_2|q)_k\ldots(\alpha_{p+1}|q)_k}{(\beta_1|q)_k\ldots(\beta_p|q)_k(q|q)_k}x^k \tag{3.10}$$

(with $x=q^{\pm(c+1)}$, $c=\sum_{i=1}^{p+1}\alpha_i-\sum_{j=1}^{p}\beta_j$ in the sums which appear in the coupling or recoupling coefficients and $c=-1$ for the balanced series) instead of the standard basic hypergeometric function (see Gasper and Rahman [29], Groza *et al* [1], Kachurik and Klimyk [2])

$${}_{p+1}\phi_p\left[\begin{matrix}q^{\alpha_1},q^{\alpha_2},\ldots,q^{\alpha_{p+1}}\\ q^{\beta_1},\ldots,q^{\beta_p}\end{matrix};q,z\right]={}_{p+1}\Phi_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1}\\ \beta_1,\ldots,\beta_p\end{matrix};q,z\right] \tag{3.11a}$$
$$={}_{p+1}F_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1}\\ \beta_1,\ldots,\beta_p\end{matrix};q^{1/2},q^{(c-1)/2}z\right]. \tag{3.11b}$$

In the series of the type (3.10) which appear in the coupling and recoupling coefficients of $\mathrm{u}_q(n)$, usually $x=q^{\pm(c+1)}$.

The second sum on the right-hand side of (3.8) corresponds to the very well-poised ${}_{10}\phi_9$ basic (or classical ${}_9F_8(1)$ in the SU(3) and $q=1$ case) hypergeometric series of the type

$${}_{2r+2}\phi_{2r+1}\left[\begin{matrix}q^{-2p_1-1},q^{-p_1+1/2},-q^{-p_1+1/2},q^{-p_1-p_2-1},\ldots,q^{-p_1-p_{2r}-1}\\ q^{-p_1-1/2},-q^{-p_1-1/2},q^{p_2-p_1+1},\ldots,q^{p_{2r}-p_1+1}\end{matrix};q,q^{\sum_i p_i+2r-1}\right]$$
$$=\frac{\prod_{i=r'+1}^{r'+r''}[p_i-p_1]![p_i+p_1+1]!}{[-2p_1-1]!\prod_{i=r'+r''+1}^{2r}[p_1-p_i-1]![-p_i-p_1-2]!}$$

$$\times \prod_{i=2}^{r'} \frac{[p_i - p_1]!}{[-p_1 - p_i - 2]!} \sum_j (-1)^{r'(p_1+j+1)} [2j+1]$$
$$\times \frac{\prod_{i=1}^{r'} [j - p_i - 1]! \prod_{i=r'+r''+1}^{2r} ([j - p_i - 1]! [-j - p_i - 2]!)}{\prod_{i=1}^{r'+r''} [p_i + j + 1]! \prod_{i=r'+1}^{r'+r''} [p_i - j]!}$$
$$p_1 \leqslant p_2, \ldots, p_{r'} \leqslant -p_1 - 2 < \min(p_{r'+1}, \ldots, p_{r'+r''}) \leqslant -p_{r'+r''+i} - 2 \tag{3.12}$$

again depending on eight free parameters

$$p_1 = \tfrac{1}{2}(b - b' - v) - \tilde{I} - 1 \qquad p_2 = \tilde{z}' - 1 \qquad p_3 = \tfrac{1}{2}(a'' - a) - 1$$
$$p_4 = \tfrac{1}{2}(a - a'') - s - 1 \qquad p_5 = \tfrac{1}{2}(b - b' - v) + \tilde{I} \qquad p_6 = \tfrac{1}{2}(a'' + a)$$
$$p_7 = a' + \tilde{z}' \qquad p_8 = b' - \tilde{z}' \tag{3.13}$$

with $r' = 4$ or 5 and $r'' = 4$ or 3 and $p_1 + j + 1$ used as the expansion parameter k of ${}_{10}\phi_9$. In particular, the left-hand side of (3.6) corresponds to a special case of (3.12) with $r = 2$. For $\tilde{I} = \tilde{z}$ or $\frac{1}{2}(b'' - a')$ the summation parameter s is fixed in (3.8) and the second sum turns into the very well-poised ${}_8\phi_7$ which again may be transformed into balanced ${}_4\phi_3$ series, in accordance with the Whipple [35] and Watson [36] transformation formula (see (2.5.1) of [29]).

When the number of values accepted by the multiplicity labels $\tilde{J}$ or $\tilde{I}'$ does not exceed the multiplicity r of irrep $(a\ b)$ in the coproduct $(a'b') \otimes (a''b'')$ decomposition, the inverse overlap matrix $(\eta^{-,+,\tilde{J}}|\eta^{\tilde{I}',-,-})_q$, which satisfies the biorthogonality relations

$$\sum_{\tilde{I}} (\eta_{\tilde{J}',-,-}|\eta_{-,+,\tilde{I}})_q (\eta^{-,+,\tilde{I}}|\eta^{\tilde{I}',-,-})_q = \delta_{\tilde{I}',\tilde{J}'} \tag{3.14a}$$
$$\sum_{\tilde{I}'} (\eta^{-,+,\tilde{I}}|\eta^{\tilde{I}',-,-})_q (\eta_{\tilde{I}',-,-}|\eta_{-,+,\tilde{J}})_q = \delta_{\tilde{I},\tilde{J}} \tag{3.14b}$$

may be introduced. Particularly, when r is determined by an entree of the left column of array (2.8b), i.e. for $b'' - a' + v \geqslant 0$ and $a'' - a - v \geqslant 0$, the analytical inversion substitution (cf Ališauskas [26]†).

$$(a', b') \to (-a' - 2, -b' - 2) \qquad (a'', b'') \to (-a'' - 2, -b'' - 2)$$
$$(a, b) \to (a - 2, -b - 2) \qquad \tilde{I}' \to \tilde{I}' \qquad \tilde{I}' \to \tilde{I}'$$
$$v \to -v \qquad \tilde{z} \to -\tilde{z} \qquad \tilde{z}' \to -\tilde{z}' \tag{3.15}$$

into (3.7), together with the change of the summation parameter $s \to s' - a - 1$ and some q-dimensional factor, leads to the expression of the inverse overlap matrix

$$(\eta^{-,+,\tilde{J}}|\eta^{\tilde{I}',-,-})_q = (-1)^{\tilde{I}'+\tilde{z}'} q^{-Q_1(a'b'a''b''ab;\tilde{J}\tilde{z}) - Q_1(abb''a''a'b';\tilde{I}'\tilde{z}') + a''/2 + b''} [2\tilde{J} + 1]$$
$$\times ([a+1][b+1][a+b+2][b']![a'+b'+1]![a'']![b'']!)^{-1/2} \nabla[\tfrac{1}{2}a'', \tfrac{1}{2}a, \tilde{I}']$$
$$\times \frac{\nabla[\frac{1}{2}b'', \frac{1}{2}a', \tilde{J}] H[a'b'\tilde{I}'\tilde{z}'] H[ab\tilde{J}\tilde{z}]}{[a'' + b'' + 1]!} \left(\frac{[2\tilde{I}' + 1][\tilde{I}' + \tilde{z}']![\tilde{J} + \tilde{z}]!}{[\tilde{I}' - \tilde{z}']![\tilde{J} - \tilde{z}]!} \right)^{1/2}$$
$$\times \sum_{s'} \frac{[a - s']![a + v - s']!}{[s']![a + \tilde{z} - \tilde{J} - s']![a + \tilde{z} + \tilde{J} - s' + 1]![\frac{1}{2}(a + a'') - \tilde{I}' - s']!}$$
$$\times \frac{[b' + b'' + v + s' + 2]!}{[\frac{1}{2}(a + a'') + \tilde{I}' - s' + 1]![a' - a - v + s']![b' - a'' + v + s']!}. \tag{3.16}$$

† Note that in addition to the corrections presented in the footnote of [28], the numerator factor $(a + b'' + v + 2)!$ of (3.6) of [26] should be corrected into $(a + b' + v + 2)!$ and $\bar{i}_z$ should be replaced by $\tilde{i}_z$ in (A1).

For $b'' - a' + v \geqslant 0$ and for $a'' - a - v \geqslant 0$, respectively, we may prove the biorthogonality relations (3.14a) and (3.14b) with inserted expressions (3.7) and (3.16), after using the summation formula

$$\sum_j \frac{(-1)^{p_1+j+1}[2j+1][j-p_1-1]!}{[p_1+j+1]![p_2-j]![p_2+j+1]!} = \delta_{p_1+p_2+1,0} \tag{3.17}$$

with $p_1 \leqslant -1$, (cf special very well-poised basic hypergeometric series ${}_4\phi_3$ of the type (3.12)–(2.3.4) of [29]) and obtain a dependence between the summation parameters $a - s - s' = 0$. Then, as the last step, for $\tilde{J}' > \tilde{I}'$ or $\tilde{J} > \tilde{I}$ we use the Gauss–Heine summation formula

$$\sum_s (-1)^s q^{s(b+c-a-1)} \frac{[a-s]!}{[s]![b-s]![c-s]!} = q^{bc} \frac{[a-b]![a-c]!}{[b]![c]![a-b-c]!} \tag{3.18}$$

(cf basic hypergeometric series ${}_2\phi_1$—(1.5.1) of [29]), when for $\tilde{J}' \leqslant \tilde{I}'$ or $\tilde{J} \leqslant \tilde{I}$ relations (3.14a) and (3.14b) may be verified straightforwardly. Hence, we see that the number of values accepted by $\tilde{J}$ in (3.14a) or by $\tilde{I}'$ in (3.14b) may exceed r separately. Unfortunately, for

$$a'' - a - v < 0 \qquad \text{and} \qquad b'' - a' + v < 0 \tag{3.19}$$

both together, equation (3.16) cannot be used (since $p_1 \geqslant 0$) even if $\tilde{J}$ and $\tilde{I}'$ both accept the same number of values coinciding with r. The sense of (3.16) in this region for the representation theory of $\mathrm{u}_q(3)$ is not clear, as well as the sense of the sum

$$\sum_s \frac{[\tilde{J} - \tilde{z} + s]![\frac{1}{2}(a - a'') + \tilde{I}' + s]!}{[s]![\tilde{J} + \tilde{z} - s]![\frac{1}{2}(a'' - a) + \tilde{I}' - s]![a+1+s]!} \times \frac{[b' + b'' + a + v + s + 3]!}{[a' - v + 1 + s]![b' - a'' + a + v + s]![-v + s]!} \tag{3.20}$$

which appears after the substitution of (3.15) into (3.7) without the change $s \rightarrow s' - a - 1$ and corresponds to an alternative summation interval of formal series.

Now we see that the sum on the right-hand side of (3.16) corresponds to the balanced (Saalschützian) ${}_5\phi_4$ basic (or ${}_5F_4(1)$ classical) hypergeometric series, depending on eight free parameters, which for $a'' - a - v \geqslant 0$ or $b'' - a' + v \geqslant 0$ may be written again in terms of series (3.10)

$$\frac{[A_3' - 1]![N_1']![N_2']!}{[n']![m']![A_1' - n']![A_2' - m']![B_1' - 1]![B_2' - 1]!} \times {}_5F_4 \left[\begin{matrix} -n', -m', A_1' + n', A_2' + m', -A_3' \\ -N_1', -N_2', B_1', B_2' \end{matrix} ; q, 1 \right] \tag{3.21a}$$

with

$$n' = a + \tilde{z} - \tilde{J} \qquad m' = \tfrac{1}{2}(a + a'') - \tilde{I}' \qquad A_1' = 2a + 2\tilde{z} + 1$$
$$A_2' = a + a'' + 1 \qquad A_3' = b' + b'' + v + 2 \qquad N_1' = a \qquad N_2' = a + v$$
$$B_1' = a' - a - v + 1 > 0 \qquad B_2' = b' - a'' + v + 1 > 0. \tag{3.21b}$$

(Otherwise the summation parameter $s'' = a' - a - v + s'$ or $b' - a'' + v + s'$ should be used instead of s'.)

Thus for $n \leqslant N_1$, $m \leqslant N_1$, $A_1 - B \geqslant 0$, $A_2 - B \geqslant 0$ integers and $N_1 + N_2 + N_3 + A_1 + A_2 - A_3 - B + 1 = 0$ we get the biorthogonal system of functions

$$R^{(3)}_{n,m} = {}_5F_4 \left[\begin{matrix} -n, -m, A_1 + n, A_2 + m, -A_3 \\ -N_1, -N_2, -N_3, B \end{matrix} ; q, 1 \right] \frac{[A_1 + n - 1]![A_2 + m - 1]!}{[n]![m]!} \tag{3.22a}$$

and

$$S^{(3)}_{m,n} = {}_5F_4\left[\begin{matrix} -N_1+n, -N_1+m, -A_1-N_1-n, -A_2-N_1-m, A_3-N_1+1 \\ -N_1, -N_1-B, N_2-N_1+1, N_3-N_1+1 \end{matrix}; q, 1\right] \times \frac{(-1)^{m+n}[A_1+2n-1][A_2+2m-1]}{[N_1-n]![N_1-m]![A_1+N_1+n]![A_2+N_1+m]!} \tag{3.22b}$$

which satisfy the biorthogonality relations

$$\sum_m R^{(3)}_{n,m} S^{(3)}_{m,n'} = \delta_{n,n'} \frac{[A_3]![B-1]![N_2-N_1]![N_3-N_1]!}{[N_1]![N_2]![N_3]![N_1]![N_1+B-1]![A_3-N_1]!} \tag{3.23a}$$

for $A_2 - B \geqslant 0$ and

$$\sum_n S^{(3)}_{m,n} R^{(3)}_{n,m'} = \delta_{m,m'} \frac{(-1)^{N_1}(-A_3|q)_{N_1}}{([N_1]!)^2(-N_2|q)_{N_1}(-N_3|q)_{N_1}(B|q)_{N_1}} \tag{3.23b}$$

for $A_1 - B \geqslant 0$ (with the same positive norm written in the right-hand side of (3.23a) and (3.23b) in different forms).

At last and at least for $a'' - a - v \geqslant 0$, we may write an expression for the overlap of the coupled states (with superscript)

$$\begin{aligned}
(\eta^{-,+,\tilde{I}}|\eta^{-,+,\tilde{J}})_q
&\equiv \sum_{y',y'',i',i''} \left[\begin{matrix} (a'b') & (a''b'') & {}^{-,+,\tilde{I}}(ab) \\ y'i' & y''i'' & yi \end{matrix}\right]^{(3)}_q \left[\begin{matrix} (a'b') & (a''b'') & {}^{-,+,\tilde{J}}(ab) \\ y'i' & y''i'' & yi \end{matrix}\right]^{(3)}_q \\
&= \sum_{\tilde{I}'} (\eta^{-,+,\tilde{I}}|\eta_{\tilde{I}',-,-})_q (\eta^{\tilde{I}',-,-}|\eta^{-,+,\tilde{J}})_q \\
&= \frac{q^{-Q_1(a'b'a''b''ab;\tilde{I}\tilde{z})-Q_1(a'b'a''b''ab;\tilde{J}\tilde{z})}[2\tilde{I}+1][2\tilde{J}+1]}{[a+1][b+1][a+b+2][b']![a'+b'+1]![a'']![a''+b''+1]!} \\
&\quad\times \frac{\nabla[\frac{1}{2}b'', \frac{1}{2}a', \tilde{J}]H[ab\tilde{J}\tilde{z}]}{\nabla[\frac{1}{2}b'', \frac{1}{2}a', \tilde{I}]H[ab\tilde{I}\tilde{z}]} \left(\frac{[\tilde{I}-\tilde{z}]![\tilde{J}+\tilde{z}]!}{[\tilde{I}+\tilde{z}]![\tilde{J}-\tilde{z}]!}\right)^{1/2} \\
&\quad\times \sum_{s',\tilde{i}'} \frac{[a-s']![a+v-s']![b'+b''+v+s'+2]!}{[s']![a+\tilde{z}-\tilde{J}-s']![a+\tilde{z}+\tilde{J}-s'+1]![\frac{1}{2}(a+a'')-\tilde{i}'-s']!} \\
&\quad\times \frac{[2\tilde{i}'+1][\tilde{i}'+\tilde{z}']!H^2[a'b'\tilde{i}'\tilde{z}']\nabla^2[\frac{1}{2}a'', \frac{1}{2}a, \tilde{i}']}{[\frac{1}{2}(a+a'')+\tilde{i}'-s'+1]![a'-a-v+s']![b'-a''+v+s']![\tilde{i}'-\tilde{z}']!} \\
&\quad\times \frac{[\tilde{I}+\tilde{i}'+\frac{1}{2}(b-b'-v)]!(b-b'-v|q)_{(b'-b+v)/2-\tilde{i}'+\tilde{I}}}{[\frac{1}{2}(b'-b+v)+\tilde{I}-\tilde{i}']![\frac{1}{2}(b'-b+v)+\tilde{I}+\tilde{i}'+1]!}.
\end{aligned} \tag{3.24}$$

Again, the second sum over $\tilde{i}'$ on the right-hand side of (3.24) corresponds to the very well-poised ${}_{10}\phi_9$ basic (or classical ${}_9F_8(1)$ in the SU(3) and $q = 1$ case) hypergeometric series of the type (3.12), depending on eight free parameters

$$p'_1 = \tfrac{1}{2}(b'-b+v)+\tilde{I} \qquad p'_2 = -\tilde{z}'-1 \qquad p'_3 = \tfrac{1}{2}(a-a'')-1$$
$$p'_4 = \tfrac{1}{2}(a''+a)-s' \qquad p'_5 = \tfrac{1}{2}(b'-b+v)-\tilde{I}$$
$$p'_6 = -\tfrac{1}{2}(a''+a)-2p'_7 = -a'-\tilde{z}'-2 \qquad p'_8 = -b'+\tilde{z}'-2 \tag{3.25}$$

with $r' = 2$ and $r'' = 3$ or 2, but with the different ordering of the parameters (with $p'_1 \geqslant 0$ and $p'_1 - j$ used as the expansion parameter k of ${}_{10}\phi_9$) to compare with the (3.12) case. Again we have a single sum for $b - b' - v = 0$, i.e. for equal the middle and right columns of array (2.8b), but additional restrictions of the second sum appear for $b - b' - v < 0$ in contrast with the (3.8) case. The summation parameter s' is fixed for $\tilde{J} = a + \tilde{z}$ or $b - \tilde{z}$ or $\frac{1}{2}(a' + b'')$ and again the second sum may be rearranged into balanced ${}_4\phi_3$ series.

4. Elementary overlaps related to the q-Racah coefficients and some canonical isofactors with the minimal null space

The recoupling coefficients of $\mathrm{u}_q(2)$ (q-Racah coefficients or q $6j$-coefficients), related to the balanced basic hypergeometric functions ${}_4\phi_3(q)$ (see Kachurik and Klimyk [2], Rajeswari and Srinivasa Rao [3]) and to the well-poised basic hypergeometric functions ${}_8\phi_7(q)$ (in accordance with the Watson [36] transformation) also appear in the mutual expansion coefficients of some definite systems of $\mathrm{u}_q(3)$ isofactors.

Equation (2.11) with inserted (2.12a) and (2.12b) and extreme values of parameters for $b'-b+v \geqslant 0$ has the summation parameter $j_2 = 0$ and the remaining summation parameters fixed and gives expression for overlaps of the coupled $\mathrm{u}_q(3)$ states which correspond to the different families of the paracanonical coupling coefficients in terms of the q-Racah coefficients (in accordance with the Regge type symmetry of the extreme first isofactor in the right-hand side of (2.11)—see Ališauskas and Smirnov [8])

$$
\begin{aligned}
(\eta_{+,-,\tilde{j}}|\eta^{-,+,\tilde{I}})_q &\equiv \begin{bmatrix} (a'b') & (a''b'') & {}^{-,+,\tilde{I}}(ab) \\ \bar{y}'_0\bar{i}'_0 & y''_0 i''_0 & \tilde{Y}\,\tilde{j} \end{bmatrix}^{(3)}_q \\
&= q^{(a-b)(a'-b'+a''-b'')/2 - Q_1(a'b'a''b''ab;\tilde{I}\tilde{z}) - Q_1(a''b''a'b'ab;\tilde{j}\tilde{Z})} \\
&\quad \times [2\tilde{I}+1]\left(\frac{[\frac{1}{2}(a''+b')+\tilde{j}+1]![\frac{1}{2}(a''+b')-\tilde{j}]![a']![b'']!}{[\frac{1}{2}(a'+b'')+\tilde{I}+1]![\frac{1}{2}(a'+b'')-\tilde{I}]![a'']![b']!}\right)^{1/2} \\
&\quad \times (-1)^{a'+b''+(a''+b')/2+\tilde{j}} \begin{Bmatrix} \frac{1}{2}(a'-b''+b-v) & \frac{1}{2}(b-v) & \hat{I} \\ \frac{1}{2}(a''-b'+b-v) & \frac{1}{2}a & \tilde{j} \end{Bmatrix}_q .
\end{aligned} \tag{4.1}
$$

Equation (4.1) is also valid for $a'' - a - v \geqslant 0$. Otherwise, the inverse expansion coefficients—overlaps $(\eta_{-,+,\tilde{I}}|\eta^{+,-,\tilde{j}})_q$ for $a' - a - v \geqslant 0$ or $b'' - b + v \geqslant 0$ may be derived from (4.1) using the substitution

$$
\begin{aligned}
&(a'b') \to (b'a') \qquad (a''b'') \to (b''a'') \qquad (ab) \to (ba) \qquad v \to -v \\
&\tilde{j} \leftrightarrow \tilde{I} \qquad q \to q^{-1}
\end{aligned} \tag{4.2}
$$

together with the corresponding phase factor (see [10]), i.e. they include the same q $6j$-coefficient, factor $[2\tilde{j}+1]$ instead of $[2\tilde{I}+1]$ and an inverted remaining factor. These non-unitary expansion coefficients are correlated with the unitary transition (Weyl) coefficients between U and T bases (Smirnov and Malashin [37], Malashin *et al* [38], Asherova *et al* [39]) for all three entrees of the coupling coefficients

$$
\begin{bmatrix} (a'b') & (a''b'') & {}^{\rho}(ab) \\ y'_0 i'_0 i'_0 & \bar{y}''_0\bar{i}''_0, -\bar{i}''_0 & \tilde{y}\tilde{I}, i'_0 - \bar{i}''_0 \end{bmatrix}^{(3)}_q \qquad \text{and} \qquad \begin{bmatrix} (a'b') & (a''b'') & {}^{\rho}(ab) \\ \bar{y}'_0\bar{i}'_0, -\bar{i}'_0 & y''_0 i''_0 i''_0 & \tilde{Y}\tilde{j}, i''_0 - \bar{i}'_0 \end{bmatrix}^{(3)}_q \tag{4.3}
$$

(i.e. identities for the HWS or LWS and the $\mathrm{u}_q(2)$ recoupling coefficients in the general case), but an additional q-phase is also necessary, since the expressions for (3.2) are not invariant under substitution $q \leftrightarrow q^{-1}$. For $b' - b + v < 0$ and $a'' - a - v < 0$ both together, overlap $(\eta_{+,-,\tilde{j}}|\eta^{-,+,\tilde{I}})_q$ gives the expansion coefficients of the coupled states $|\eta_{+,-,\tilde{j}})_q$ in terms of the overcomplete states $|\eta_{-,+,\tilde{I}})_q$, as well as $(\eta_{-,+,\tilde{I}}|\eta^{+,-,\tilde{J}})_q$ for $a'-a-v<0$ and $b''-b+v<0$ gives the expansion coefficients of the coupled states $|\eta_{-,+,\tilde{I}})_q$ in terms of the overcomplete states $|\eta_{+,-,\tilde{J}})_q$, which should be expanded in terms of the complete basis. Hence, in these regions the problem of inversion of the truncated $u_q(2)$ recoupling matrix appears (cf the SU(2) case, [26]).

The same equation (2.11) with inserted (2.13*a*) and (2.12*b*) and extreme values of parameters for $b-b'-v\geqslant 0$ or $b''-a'+v\geqslant 0$ gives the boundary isofactors which allow (in terms of isofactors related to (2.9), but with the subscript $_{\tilde{I}',-,-}$) the bilinear combinations of isofactors cover with the MIS, which are characterized by parameters (2.7*c*) and correlated (when $b-b''-v\geqslant 0$ and $a-a''+v\geqslant 0$ see [10, 26]) with the canonical coupling scheme of Biedenharn and Louck [17, 40]:

$$
\begin{aligned}
(\eta_{\hat{J}',\uparrow,\uparrow}|\eta^{\tilde{I}',-,-})_q &= \begin{bmatrix} (a'b') & (a''b'') & {}^{\tilde{I}',-,-}(ab) \\ \hat{y}'\hat{J}' & y''_{\mathrm{m}}i''_{\mathrm{m}} & y_{\mathrm{m}}i_{\mathrm{m}} \end{bmatrix}^{(3)}_q \\
&= \left(\frac{[2\hat{J}'+1][2\tilde{I}'+1][a'']![a+1]![i_{\mathrm{m}}+i''_{\mathrm{m}}-\hat{J}']![i_{\mathrm{m}}+i''_{\mathrm{m}}+\hat{J}'+1]!}{[\frac{1}{2}(a''+a)-\tilde{I}']![\frac{1}{2}(a''+a)+\tilde{I}'+1]![a''+b'']![a+b+1]!}\right)^{1/2} \\
&\quad\times(-1)^{a''-b''-a+b}q^{Q_2}\begin{Bmatrix} \frac{1}{2}(b'+b''-b+v) & \frac{1}{2}(b'-a''+a+v) & \hat{J}' \\ \frac{1}{2}(b'+v) & \frac{1}{2}a' & \tilde{I}' \end{Bmatrix}_q
\end{aligned}
\qquad (4.4)
$$

where

$$
\begin{aligned}
Q_2 = {} & \tfrac{1}{8}(a-a'')(a-a''+2)-\tfrac{3}{2}\tilde{I}'(\tilde{I}'+1)-\tfrac{1}{2}\hat{J}'(\hat{J}'+1) \\
& +\tfrac{1}{2}(i_{\mathrm{m}}-i''_{\mathrm{m}})(i_{\mathrm{m}}-i''_{\mathrm{m}}+1)+\tfrac{1}{2}a'(a'+1)-(a'+\tilde{z}')^2 \\
& +\tfrac{1}{2}(b'+b''-b+v)(2a'+b'-b''+b-v+3).
\end{aligned}
\qquad (4.5)
$$

We note from [10] that the canonical $u_q(3)$ isofactors are determined (under the definite restrictions) by the vanishing condition of the extreme isofactors

$$
\begin{bmatrix} (a'b') & \overset{t}{(a''b'')} & (ab) \\ y'i' & y''_{\mathrm{m}}i''_{\mathrm{m}} & yi \end{bmatrix}^{(3)}_q \qquad (4.6a)
$$

with parameters

$$
|i-i'|>i''_{\mathrm{m}}-t+1 \qquad (4.6b)
$$

where isospin accepts the special values $i''_{\mathrm{m}}\equiv\frac{1}{2}(a''+b'')$ and $y''_{\mathrm{m}}=\frac{1}{3}(a''-b'')$. Hence, $\hat{J}'$ is correlated in (4.4) with the multiplicity label $t=\hat{J}'+i''_{\mathrm{m}}-i_{\mathrm{m}}+1$ of the canonical tensor operator $T^{(a''b'')t,q}_{y''i''i''_z}$ if the extreme isofactor

$$
\begin{bmatrix} (a'b') & \overset{t=1}{(a''b'')} & (ab) \\ \hat{y}'i_{\mathrm{m}}-i''_{\mathrm{m}} & y''_{\mathrm{m}}i''_{\mathrm{m}} & y_{\mathrm{m}}i_{\mathrm{m}} \end{bmatrix}^{(3)}_q \qquad (4.6c)
$$

does not vanish. Particularly, the coupled non-orthonormal state $|\eta_{\hat{J}',\uparrow,\uparrow})_q$ includes the canonical coupled states with $t\leqslant\hat{J}'+i''_{\mathrm{m}}-i_{\mathrm{m}}+1$. The multiplicity label t accepts values $1,2,\ldots,\mathcal{M}$, where $\mathcal{M}$ is the number of independent canonical tensor operators

$$
\mathcal{M}=\min r_{\alpha'\beta'\gamma}+1 \qquad (\alpha'=2,3;\ \beta'=2,3;\ \gamma=1,2) \qquad (4.7)
$$

characterized by the $2\times2\times2$ subarray of (2.8*b*).

For the external multiplicity r determined by $a'+a''-a-v$ or $b'+b''-b-v\leqslant r_{\alpha\beta\gamma}$ from array (2.8*b*), the overlap matrix $(\eta_{\hat{J}',\uparrow,\uparrow}|\eta^{\tilde{I}',-,-})_q$ may be inverted and $(\eta_{\tilde{I}',-,-}|\eta^{\hat{J}',\uparrow,\uparrow})_q$

may be expressed in terms of the same recoupling matrix (q $6j$-coefficients with factor $(-1)^{a''+b''+a+b}([2\hat{J}'+1][2\tilde{I}'+1])^{1/2}$), with the remaining factors in the right-hand side of (4.4) inverted. In this case the expansion coefficients (4.4) in terms of the $\mathrm{u}_q(2)$ recoupling coefficients are correlated with the transition coefficients between V and U bases (or V and T bases, see Malashin *et al* [38]) for all three entrees of some coupling coefficients (i.e. the identities for the HWS, LWS or MIS and the $\mathrm{u}_q(2)$ recoupling coefficients with braiding [41] in the general case), but an additional q-phase is also necessary, in addition to the braiding q-phase of the type $q^{\pm\tilde{I}'(\tilde{I}'+1)}$ which appears, for example, in the composition of $(\eta^{-,+,\tilde{J}}|\eta^{\tilde{I}',-,-})_q$ and $(\eta_{\tilde{I}',-,-}|\eta^{\hat{J}',\uparrow,\uparrow})_q$, in contrast with the compositions of overlaps presented by (4.1) and in section 3, where such a q-phase is eliminated. In this case the coupled non-orthonormal state $|\eta^{\hat{J}',\uparrow,\uparrow})_q$ includes the canonical coupled states with $t \geqslant \hat{J}' + i''_{\mathrm{m}} - i_{\mathrm{m}} + 1$ and the state with maximal value of $\hat{J}' = \frac{1}{2}(a'+b'-|v|)$ leads to the canonical isofactor with the maximal value of t.

For the highest weight component of tensor $T^{(a''b'')t,q}_{y''_0 i''_0 i''_{z0}}$ we may derive the extreme isofactors

$$\begin{aligned}
&\left[\begin{matrix} (a'b') & (a''b'') & {}^{\hat{J}',\uparrow,\uparrow}(ab) \\ y'i' & y''_0 i''_0 & yi \end{matrix}\right]^{(3)}_q \\
&\quad = \sum_{\tilde{I}'} (\eta^{\hat{J}',\uparrow,\uparrow}|\eta_{\tilde{I}',-,-})_q \left[\begin{matrix} (a'b') & (a''b'') & {}^{\tilde{I}',-,-}(ab) \\ y'i' & y''_0 i''_0 & yi \end{matrix}\right]^{(3)}_q \qquad (4.8a)\\
&\quad = \frac{\Gamma[a'b'i'z']\Gamma[a'b'\hat{J}'\hat{z}']\nabla[i''_0 i i']}{\Gamma[abiz]\nabla[i''_{\mathrm{m}} i_{\mathrm{m}} \hat{J}']} \left(\frac{[2i'+1][2\hat{J}'+1][a''+b'']!}{[a'']![a]![b]!}\right)^{1/2} \\
&\qquad \times \sum_{n_1,n_2} \frac{(-1)^{v+i'+i''_0-i+n_1+n_2} q^{Q_3(\hat{J}')-(a'+a''-a-v-n_1)(i'+z'+v-n_2)}}{[n_1]![n_2]![\frac{1}{2}(a'+b'+v)-\hat{J}'-n_1]![\frac{1}{2}(a'+b'+v)+\hat{J}'-n_1+1]!} \\
&\qquad \times \frac{[a'-n_1]![a'+b'+1-n_1]![2i'-n_2]![a+z+i''_0+i'-n_2+1]!}{[i'+i''_0-i-n_2]![i'+i''_0+i-n_2+1]![a'+z'+i'-n_1-n_2+1]!} \qquad (4.8b)
\end{aligned}$$

where $\nabla[abc]$ is expressed as (3.3),

$$\tilde{z}' = \tfrac{1}{2}(a''-a)-v) \qquad \hat{z}' = \tfrac{1}{2}(b'-a'+v)$$

$$\Gamma[abiz] = \left(\frac{[i+z]![a+z-i]![a+z+i+1]!}{[i-z]![b-z-i]![b-z+i+1]!}\right)^{1/2} \qquad (4.9)$$

and

$$\begin{aligned}
Q_3(\hat{J}') = \tfrac{1}{2}\{&(a+2z)(\tfrac{1}{2}a''+b'') - (b'+b''-b+v)(2a'+b'-b''+b-v+3) \\
&+ i_0(i_{0+1}) - i(i+1) + i'(i'+1) + \hat{J}'(\hat{J}'+1) - (i_{\mathrm{m}}-i''_{\mathrm{m}})(i_{\mathrm{m}}-i''_{\mathrm{m}}+1) \\
&+ (i_0-i''_0)(i_0-i''_0+1) - a'(a'+1)\} + (a'+\tilde{z}')^2. \qquad (4.10)
\end{aligned}$$

In order to derive (4.8b), we expressed special isofactor with superscript $\tilde{I}', -, -$ in (4.8a) by means of (4.6) of [9], using the symmetry relations of the type (4.5b) and (4.2a) of [10]. The q $6j$-coefficients which appeared in both the factors of (4.8a), were expressed by means of the q-analogue of the less symmetric expression (29.1c) of Jucys and Bandzaitis [42] (see (5.22) of Asherova *et al* [39]) with the triangles $\frac{1}{2}a+z, i', \tilde{I}'$ and $\frac{1}{2}a'+\tilde{z}', \frac{1}{2}a', \tilde{I}'$ instead of parameters f, b, d of the $6j$-coefficients, respectively. After using the summation formula

$$\sum_j \frac{(-1)^{p_1+j+1} q^{j(j+1)-p_1(p_1+1)}[2j+1][j-p_1-1]!}{[p_1+j+1]![p_2-j]![p_2+j+1]![p_3-j]![p_3+j+1]!} = \frac{q^{(p_1+p_2+1)(p_1+p_3+1)}}{[p_1+p_2+1]![p_1+p_3+1]![p_2+p_3+1]!} \tag{4.11}$$

considered in the next section, we obtain that both the remaining sums in (4.8*b*) are of the ${}_3\phi_2$ type and depend on nine parameters both together (from ten independent parameters in the left-hand side).

For $v \leqslant 0$ and $\hat{J}'_{\max} = \frac{1}{2}(a'+b'+v)$ we have ratio of the minimal null space case isofactors

$$\begin{bmatrix} (a'b') & \overset{t_{\max}}{(a''b'')} & (ab) \\ y'i' & y''_0 i''_0 & yi \end{bmatrix}_q^{(3)} \left(\begin{bmatrix} (a'b') & \overset{t_{\max}}{(a''b'')} & (ab) \\ \hat{y}'\hat{J}'_{\max} & y''_{\mathrm{m}} i''_{\mathrm{m}} & y_{\mathrm{m}} i_{\mathrm{m}} \end{bmatrix}_q^{(3)} \right)^{-1} \tag{4.12a}$$

with $t_{\max} = a' + a'' - a - v + 1$, instead of the left-hand side of (4.8*a*), fixed summation parameter $n_1 = 0$ and the remaining sum of the ${}_3\phi_2$ type in the right-hand side Otherwise, for $v > 0$ and $\hat{J}'_{\max} = \frac{1}{2}(a'+b'-v)$ we use (3.18) and express the isofactors with the minimal null space as follows:

$$\begin{aligned} &\begin{bmatrix} (a'b') & \overset{t_{\max}}{(a''b'')} & (ab) \\ y'i' & y''_0 i''_0 & yi \end{bmatrix}_q^{(3)} \left(\begin{bmatrix} (a'b') & \overset{t_{\max}}{(a''b'')} & (ab) \\ \hat{y}'\hat{J}'_{\max} & y''_{\mathrm{m}} i''_{\mathrm{m}} & y_{\mathrm{m}} i_{\mathrm{m}} \end{bmatrix}_q^{(3)} \right)^{-1} \\ &\quad = q^{Q_3(\hat{J}'_{\max})-(i'+z')(b'+b''-b+v)+v(a-a''+v+1)} \frac{\Gamma[a'b'i'z']\nabla[i''_0 i i']}{\Gamma[abiz]([a'']![a]![b]![v]!)^{1/2}} \\ &\qquad \times \left(\frac{[2i'+1][a''+b'']![a'+b'+1]![a'-v]![b']![b'-a''+a+v]!}{[a'+b'-v]![a''-b'+b-v]![b'+b''-b+v]![a+b'+b''+v+1]!} \right)^{1/2} \\ &\qquad \times \sum_{n_2} \frac{(-1)^{i'+i''_0-i+n_2} q^{(b'+b''-b+v)n_2}[v+i'+z'-n_2]![2i'-n_2]!}{[n_2]![i'+z'-n_2]![i'+i''_0-i-n_2]![i'+i''_0+i-n_2+1]!} \\ &\qquad \times \frac{[a+v+z'+i'-n_2+1]!}{[a'+z'+i'-n_2+1]!} \end{aligned} \tag{4.12b}$$

with the single sum of the non-balanced ${}_4\phi_3$ type and $t_{\max} = b' + b'' - b + v + 1$.

In the case of the self-adjoint canonical tensor operator of rank (kk) with the maximal value of $t = k+1$ from (4.12*a*) or (4.12*b*) using (3.18) we obtain expression for the special isofactor

$$\begin{bmatrix} (ab) & \overset{t=k+1}{(kk)} & (ab) \\ y'i' & y''_0 i''_0 & yi \end{bmatrix}_q^{(3)} = \frac{q^{3(i'-i)(i'+i+1)/2+k(k-6)/8+kz/2}([2i'+1])^{1/2}\Gamma[abi'z']}{\mathcal{D}\left(\begin{smallmatrix} q,t=k+1 \\ kk \end{smallmatrix}\right)[ab;ab]\nabla[i''_0 i i']\Gamma[abiz]} \tag{4.13}$$

where $i''_0 = -z''_0 = \frac{1}{2}k$, when the extreme denominator isofactor of (4.12*a*) or (4.12*b*) was expressed as special case of isofactor

$$\begin{bmatrix} (ab) & \overset{t=k+1}{(kk)} & (ab) \\ yi' & 0k & yi \end{bmatrix}_q^{(3)} = \frac{\delta_{i',i} q^{k(b-a)/6-ky}}{\mathcal{D}\left(\begin{smallmatrix} q,t=k+1 \\ kk \end{smallmatrix}\right)[ab;ab]} \left(\frac{[k]![2i+k+1]!}{[2k]![2i+1][2i-k]!} \right)^{1/2} \tag{4.14}$$

derived by means of an elementary recursive construction (cf Draayer and Akiyama [27],Irmališkas [28]), beginning from the $k = 1$ case (Smirnov and Kharitonov [7]†).

† The explicit expressions for the $u_q(3)$ canonical tensor operators are rather complicated even in the case of rank (1 1)—see Smirnov and Kharitonov [43]. Note that the substitution $q \to q^{-1}$ is necessary for the correlation of our results and isofactors presented by Smirnov *et al* [4], Smirnov and Kharitonov [6, 7], since they use T-basis.

Using the symmetry property of the isoscalar factors (see (4.5) of [10]) and the normalization condition, in some analogy with Louck *et al* [44] we obtain the square of the normalization factor

$$\mathcal{D}^2\begin{pmatrix} q, t = k+1 \\ kk \end{pmatrix}[ab; ab]$$

$$= \frac{q^{(a-b)(k+1)}[k+1][k+1]![2k+2]}{[2k+1]![a+1][b+1][a+b+2]} \sum_{i-z\geqslant 0}^{a} \sum_{i+z\geqslant 0}^{b} q^{(2k+3)2z} \frac{[2i+k+1]!}{[2i-k]!} \tag{4.15a}$$

$$= \frac{q^{(a-b)(k+1)}[k+1]![k+1][2k+2]}{[2k+3]![a+1][b+1][a+b+2]} (q^{(2k+3)(b-a)}(a+b-k+1|q)_{2k+3}$$
$$-q^{-(2k+3)(a+1)}(a-k|q)_{2k+3} - q^{(2k+3)(b+1)}(b-k|q)_{2k+3}) \tag{4.15b}$$

where the three terms of (4.15b) appeared after the double sum in (4.15a), expanded as

$$\sum_{n_1\geqslant 0, n_2\geqslant 0} \frac{q^{(2k+3)(n_2-n_1)}[n_1+n_2+k+1]![A-n_1]![B-n_2]!}{[n_1+n_2-k]![A-n_1]![B-n_2]!} \tag{4.16}$$

with $n_1 \geqslant 0, n_2 \geqslant 0$ and $A = a, B = b$, was taken using the standard Gauss–Heine summation formula

$$\sum_s q^{s(a+b-c+2)} \frac{[a-s]![b+s]!}{[s]![c-s]!} = q^{(b+1)c} \frac{[a-c]![b]![a+b+1]!}{[c]![a+b-c+1]!} \tag{4.17}$$

of terminating ${}_2\phi_1$ (cf (1.5.2) of [29]) in three regions, for $A = a, B = b$; $A = -1, B = b$ and $A = a, B = -1$, respectively, and in this way the terms with negative values of n_1 or n_2 were eliminated.

Since the matrix elements of the 'edge' components of the self-adjoint canonical tensor operator $T^{(kk)k+1,q}_{y''i''i''_z}$ are presented as (4.13) and (4.14), we may also deduce the isofactors, corresponding to the border of the weight space of irrep $(k\ k)$:

$$\begin{bmatrix} (ab) & \overset{t=k+1}{(k\ k)} & (ab) \\ (z')i' & (i''-k)i'' & (z)i \end{bmatrix}^{(3)}_q = \left(\frac{[2i'+1][2k-2i'']!}{[2i''-k]![2i'']!}\right)^{1/2}$$
$$\times \frac{q^{Q'_3}\nabla[i''i'i]\Gamma[abi'z']}{\mathcal{D}\binom{q, t=k+1}{kk}[ab; ab]\nabla^2[k-i'', i', i]\Gamma[abiz]} \tag{4.18}$$

with

$$Q'_3 = z(3i''-k) + \tfrac{1}{2}\{(a-b)(2i''-k) + (k-i'')(3i''-k-3) + 3(i'-i)(i'+i+1)\}. \tag{4.19}$$

(cf (4.4a) of [28] in the SU(3) case.) Equation (4.18) may also be proved by induction, using recursive construction.

5. Overlaps related to the canonical tensor operators

Since the results of the previous section are insufficient (incomplete) for the expansion of the coupled states associated with the canonical coupling scheme of $\mathrm{u}_q(3)$, we consider the generalization to $\mathrm{u}_q(3)$ of the Ališauskas [28] construction of the non-orthonormal SU(3) tensor operators related to the canonical tensor operators. The extreme canonical isofactors with the minimal value of multiplicity label $t = 1$

$$\begin{bmatrix} (a'b') & \overset{t=1}{(a''b'')} & (ab) \\ \bar{y}'_0\bar{i}'_0 & y''_0 i''_0 & \tilde{Y}\tilde{i} \end{bmatrix}^{(3)}_q \begin{bmatrix} (a'b') & \overset{t=1}{(a''b'')} & (ab) \\ \hat{y}' i_{\mathrm{m}} - i''_{\mathrm{m}} & y''_{\mathrm{m}} i''_{\mathrm{m}} & y_{\mathrm{m}} i_{\mathrm{m}} \end{bmatrix}^{(3)}_q \tag{5.1}$$

(and $\tilde{Z} = \frac{1}{2}(b' - a'') + v$) may be expressed by means of (3.1) of締Ališauskas [10] with the summation parameters accepting fixed values $j' = \frac{1}{2}b'$ and $m' = b - b'' - v - \frac{1}{2}b' + n_2$. The ${}_3\phi_2$ type sum over n_1 may be rearranged in a such way that the second sum accepts also the ${}_3\phi_2$ form. Both together they may be presented as

$$\sum_{u_1,u_2} \frac{(-1)^{u_1+u_2} q^{u_1(b'+b''+v+1-u_2)-(a'+a''-a-v)(b'+b''-b+v-u_2)} [a + \tilde{Z} - \tilde{i} + u_1]![b' - u_2]!}{[u_1]![u_2]![b' + b'' - b + v - u_1 - u_2]![a' + a'' - a - v - u_1]!}$$
$$\times \frac{[b'' - b + v + \frac{1}{2}(b' + a'') + \tilde{i} - u_1]![b'' + \frac{1}{2}(b' + a'') - \tilde{i} - u_2]!}{[a + b + 2 + u_1]![\frac{1}{2}(b' + a'') - \tilde{i} - u_2]![b' + b'' + v + 1 - u_2]!}$$
$$= \frac{[\frac{1}{2}(b' - a'') + \tilde{i}]![a' + a'' - v + 1]![b'']![a + \tilde{Z} - \tilde{i}]!}{[\frac{1}{2}(b' + a'') - \tilde{i}]![b - \tilde{Z} + \tilde{i} + 1]!}$$
$$\times \sum_j \frac{(-1)^{(a'+a''-a-b-v)/2+j} q^{j(j+1)-(a+b-a'-a''+v)(a+b-a'-a''+v+2)/4}}{\nabla^2[\frac{1}{2}b'', \frac{1}{2}b', j]\nabla^2[\frac{1}{2}(a' + a'' - a - v), \frac{1}{2}b, j]}$$
$$\times \frac{[2j + 1]\nabla^2[\frac{1}{2}(a'' + b''), \tilde{i}, j]}{[a + \frac{1}{2}(b' + b'') + v - j + 1]![a + \frac{1}{2}(b' + b'') + v + j + 2]!}. \tag{5.2}$$

The double sum of the left-hand side of (5.2) was replaced by a single sum using the relation

$$\sum_j \frac{(-1)^{p_1+j+1} q^{j(j+1)-p_1(p_1+1)} [2j + 1][j - p_1 - 1]![j - p_2 - 1]![j - p_3 - 1]!}{[p_1 + j + 1]![p_2 + j + 1]![p_3 + j + 1]![p_4 - j]![p_4 + j + 1]!}$$
$$\times \frac{[j - p_7 - 1]![-p_7 - j - 2]!}{[p_5 - j]![p_5 + j + 1]![p_6 - j]![p_6 + j + 1]!}$$
$$= [-p_1 - p_2 - 2]![-p_1 - p_3 - 2]![-p_2 - p_3 - 2]![-p_4 - p_7 - 2]!$$
$$\times[-p_6 - p_7 - 2]!\{[p_4 + p_6 + 1]!\}^{-1}$$
$$\times \sum_{u,v} (-1)^u q^{u(p_4+p_5+1-v)-(p_1+p_4+1-v)(p_1+p_5+1)} [p_4 - p_7 - 1 - v]!$$
$$\times\{[u]![p_1 + p_4 + 1 - u - v]![p_1 + p_5 + 1 - u]![-p_2 - p_3 - 2 - u]!$$
$$\times[p_2 - p_1 + u]![p_3 - p_1 + u]![v]![p_4 + p_5 + 1 - v]!$$
$$\times[-p_6 - p_7 - 2 - v]![p_6 - p_4 + v]!\}^{-1} \tag{5.3}$$

which is valid for non-negative arguments of q-factorials and was derived after expressing the matrix elements

$$\left\langle \begin{matrix} ab \\ YII \end{matrix} \middle| E_{13}^{\alpha} E_{32}^{\beta} \middle| \begin{matrix} ab \\ yim \end{matrix} \right\rangle_q \tag{5.4}$$

as a single sum straightforwardly using the matrix elements of the generator powers (2.17) of [10] and as double sums, after expressing the state $|abyim\rangle_q$ in terms of $E_{21}^{i-m}\,|abyii\rangle_q$, using the permutation formulae of Smirnov *et al* [4, 39], in analogy with the SU(3) case (cf Appendix A of [28]), and rearranging the separate sums, at first as a terminating balanced ${}_4\phi_3$ hypergeometric series and later as terminating ${}_3\phi_2$ series.

Although the series in the left-hand side of (5.3) resembles (3.12), the 'braiding' q-phase factor $q^{j(j+1)-p_2(p_2+1)}$ (depending on the summation parameter j) excludes the very well-poised ${}_9\phi_8$ basic hypergeometric series, but the q-versions of relations (A5*a*) and (A5*b*) of [28]† may be extended for rearrangement (without any q-phase) of the very well-poised

† Note that the factor $(p_1 - p_2 + u)!$ should appear instead of $(p_1 - p_2 + u)$ in (A5*a*) of [28] and some other corrections are presented as footnotes in [10].

${}_{10}\phi_9$ basic hypergeometric series of the type (3.12) in (3.8) and (3.24) into the double sums—compositions of balanced ${}_4\phi_3$ basic hypergeometric series. Nevertheless, the left-hand side of (5.3) for $q = 1$ corresponds to the very well-poised classical ${}_8F_7(-1)$ series. Otherwise, both the separate sums in the right-hand side are of the ${}_3\phi_2$ (or ${}_3F_2$) type. Taking $p_7 = -p_6 - 2$ we obtain the transformation formula

$$\sum_j \frac{(-1)^{p_1+j+1} q^{j(j+1)-p_1(p_1+1)}[2j+1][j-p_1-1]![j-p_2-1]![j-p_3-1]!}{[p_1+j+1]![p_2+j+1]![p_3+j+1]![p_4-j]![p_4+j+1]![p_5-j]![p_5+j+1]!}$$
$$= \frac{[-p_1-p_2-2]![-p_1-p_3-2]![-p_2-p_3-2]!}{[p_4+p_5+1]!} \sum_u \frac{(-1)^u}{[u]![p_1+p_4+1-u]!}$$
$$\times \frac{q^{u(p_4+p_5+1-v)-(p_1+p_4+1-v)(p_1+p_5+1)}}{[p_1+p_5+1-u]![-p_2-p_3-2-u]![p_2-p_1+u]![p_3-p_1+u]!} \tag{5.5}$$

resembling the Watson [36] transformation. Otherwise, the left-hand side of (5.5) corresponds to the sum which appeared in the expression for the $\mathrm{u}_q(2)$ Clebsch–Gordan coefficients, derived by Álvarez-Nodarse and Smirnov [34], when the right-hand side corresponds to the most symmetric expression of the $\mathrm{u}_q(2)$ Clebsch–Gordan coefficients (Ruegg [45]). Again, for $q = 1$ we obtain the relation between the very well-poised classical ${}_6F_5(-1)$ series and the ${}_3F_2(1)$ series

$${}_6F_5\left(\begin{matrix} a, 1+\frac{1}{2}a, b, c, d, -n \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a+n \end{matrix}; -1\right)$$
$$= \frac{(1+a)_n}{(1+a-b)_n} {}_3F_2\left(\begin{matrix} b, a-c-d+1, -n \\ 1+a-c, 1+a-d, \end{matrix}; 1\right) \tag{5.6}$$

resembling the Whipple [35] transformation.

We may also take $p_2 = -p_3 - 2$ in (5.5), relabel the parameters and obtain the summation formula (4.11), which is the q-analogue of the special summation formula of terminating ${}_4F_3(-1)$ (cf (2.3.4.8) of Slater [46]).

Let us return to our expression for (5.1). Using symmetry relation (4.2a) of [10] after inserting (3.8) of [10] for the second isofactor in (5.1), we obtain the boundary canonical isofactor, related to the overlap $(\eta_{i_{\mathrm{m}}-i''_{\mathrm{m}},\uparrow,\uparrow}|\eta_{-,+,\tilde{I}})_q$:

$$\left[\begin{matrix} (a'b') & \overset{t=1}{(a''b'')} & (ab) \\ y'_0 i'_0 & \bar{y}''_0 \bar{i}''_0 & \tilde{y}\tilde{I} \end{matrix}\right]^{(3)}_q = \frac{[a'']!([a+1][b+1][a+b+2][b']![a'+b'+1]![b'']!)^{1/2}}{\mathcal{D}\binom{q,t=1}{a''b''}[a'b';ab]\Gamma[ab\tilde{I}\tilde{z}]\nabla[\frac{1}{2}b'', \frac{1}{2}a', \tilde{I}]}$$
$$\times \sum_j \frac{(-1)^{(b''-a'')/2-\tilde{I}+j} q^{Q_4(a'b'a''b''ab)+j(j+1)+\tilde{I}(\tilde{I}+1)/2}}{\nabla^2[\frac{1}{2}a'', \frac{1}{2}a', j]\nabla^2[\frac{1}{2}(a'+a''-a)-v, \frac{1}{2}a, j]}$$
$$\times \frac{[2j+1]\nabla^2[\frac{1}{2}(a''+b''), \tilde{I}, j]}{[b-v+\frac{1}{2}(a'+a'')-j+1]![b-v+\frac{1}{2}(a'+a'')+j+2]!} \tag{5.7}$$

where

$$Q_4(a'b'a''b''ab) = (b'-v+1)(b'+b''-b+v) - \tfrac{1}{8}(a'+b'')\left(a'+b''+2\right)$$
$$-\tfrac{1}{2}\{a''(a+b'-a''+v) - b''(b''-b+v)\}$$
$$-\tfrac{1}{4}(a+b-b'-b''-v)(a+b-b'-b''-v+2) \tag{5.8}$$

$\nabla[abc]$ is defined as (3.3), $\Gamma[abiz]$ is defined as (4.9). The square of denominator function (Ališauskas [10]) may be expressed in different regions using the different symmetry relations and expressions

$$\mathcal{D}^2\begin{pmatrix} q,\, t=1 \\ a''b'' \end{pmatrix}[a'b';ab] = \mathcal{D}^2\begin{pmatrix} q^{-1},\, t=1 \\ b''a'' \end{pmatrix}[b'a';ba] \tag{5.9a}$$

$$= q^{b''(a'-b''+b-v+2)-a''(b'-a''+a+v+2)}\mathcal{D}^2\begin{pmatrix} q^{-1},\, t=1 \\ a''b'' \end{pmatrix}[ba;b'a'] \tag{5.9b}$$

$$\begin{aligned} &= \sum_{n_1,n_2} \frac{(-1)^{n_1+n_2} q^{a''n_1-b''n_2}[a+b+n_1+n_2+2]!}{[n_1]![n_2]![a+n_1+1]![b+n_2+1]![a+b+n_1+2]!} \\ &\quad\times \frac{[a-a''+v+n_1]![b'-a''+a+v+n_1+1]!}{[a'+a''-a-v-n_1]![a+b+n_2+2]!} \\ &\quad\times \frac{[b-b''-v+n_2]![a'-b''+b-v+n_2+1]!}{[b'+b''-b+v-n_2]![a+b-a''-b''+n_1+n_2+1]!} \end{aligned} \tag{5.9c}$$

$$\begin{aligned} &= (-1)^{b'-b+v} q^{a''(a''-a-v-1)-b''(b''-b+v-1)+b'-b+v} \\ &\quad\times \frac{[a'']![b'']![a'-b''+b-v+1]![b'-a''+a+v+1]![b'-a''+v]!}{[b'+b''+v+1]![a'+a''+b-v+2]![b'+b''+a+v+2]!} \\ &\quad\times \sum_{s,u} \frac{(-1)^{s+u} q^{s(b''-b+v)-u(a''+b-v+2)}[a''+b''-s-u]![s+u]!}{[s]![a-a'+b''+v-s]![b-b'+a''-v-s]![b'-b+v+s]!} \\ &\quad\times \frac{[a'-b''-v+s]!}{[a'+b'-a''-b''+s+1]![u]![a'+a''-a-v-u]![b'+b''-b+v-u]!} \\ &\quad\times \frac{[a'+b'+a''+b''-u+2]!}{[b-b'-v+u]![a'+a''-v-u+1]!} \end{aligned} \tag{5.9d}$$

which presents a rearrangement problem that is of present separate interest for investigation of the multiple basic hypergeometric series (cf Milne [47]).

Using the same equations (3.1) (as a double sum, with fixed the same summation parameters $j' = \frac{1}{2}b'$ and $m' = b - \frac{1}{2}b' - b'' - v + n_2$) and (3.8) of [10] and the q-version of Minton's summation formula

$$\sum_x \frac{(-1)^x q^{x(a+b-c)}[c-x]!}{[x]![a-x]![b-x]![S+1-x]} = (-1)^{a+b-c} q^{(a+b-c)(S+1)} \frac{[S-a]![S-b]!}{[S-c]![S+1]!} \tag{5.10}$$

(cf (1.9.6) of [29]) we may derive the boundary canonical isofactors, related to the overlap $(\eta_{i_\mathrm{m}-i''_\mathrm{m},\uparrow,\uparrow}|\eta_{+,\tilde{I}'',+})_q$:

$$\begin{aligned} &\left[\begin{matrix} (a'b') & \overset{t=1}{(a''b'')} & (ab) \\ \bar{y}'_0\bar{i}'_0 & \tilde{Y}''\tilde{i}'' & \bar{y}_0\bar{i}_0 \end{matrix}\right]^{(3)}_q \\ &\quad= \frac{([a+1]![a+b+2]![a']![a'+b'+1]![\tilde{i}''-\tilde{Z}'']!)^{1/2}}{([\tilde{i}''+\tilde{Z}'']!)^{1/2}\mathcal{D}\left(\begin{smallmatrix} q,\,t=1 \\ a''b'' \end{smallmatrix}\right)[a'b';ab]\nabla[\frac{1}{2}b,\,\frac{1}{2}b',\,\tilde{i}'']R[a''b''\tilde{i}''\tilde{Z}''][a'+a''-v+1]!} \\ &\qquad\times \sum_s \frac{(-1)^s q^{Q_5-s(a'+a''-v+1)}[\tilde{i}''+\tilde{Z}''+s]![b'+v-s]!}{[s]![\tilde{i}''-\tilde{Z}''-s]![-v+s]![a'+a''+b-v-s+2]!} \end{aligned} \tag{5.11}$$

where $\tilde{Z}'' = \frac{1}{2}(b-b') - v$,

$$R[abiz] = \left(\frac{[a+z-i]![a+z+i+1]![b-z-i]![b-z+i+1]!}{[2i+1][a]![b]![a+b+1]!}\right)^{1/2} \tag{5.12}$$

$$\begin{aligned} Q_5 &= (a'+a''+a-v)^2 + \tfrac{1}{2}\{(b'+b''-b+v)(a+b-2b''-2v)+av \\ &\quad + \tilde{i}''(\tilde{i}''+1)\} - \tfrac{1}{8}(b-b')(b-b'+2). \end{aligned} \tag{5.13}$$

We may rearrange the ${}_3\phi_2(q)$ type sum in (5.11) to a more symmetric form by analogy with the different expressions for the Clebsch–Gordan coefficients of $\mathrm{u}_q(3)$ (cf Nomura [48], Ruegg [45]) and apply the symmetry relation of the $\mathrm{u}_q(3)$ canonical isofactors (see (4.2a) of [10]). This way we can derive an expression for extreme isofactors

$$\begin{bmatrix} (a'b') & \overset{t=1}{(a''b'')} & (ab) \\ y'_0 i'_0 & \tilde{y}''\tilde{I}'' & y_0 i_0 \end{bmatrix}_q^{(3)} = \frac{(-1)^v([b+1]![a+b+2]![b']![a'+b'+1]![\tilde{I}''-\tilde{z}'']![\tilde{I}''+\tilde{z}'']!)^{1/2}}{\mathcal{D}\binom{q,t=1}{a''b''}[a'b';ab]R[a''b''\tilde{I}''\tilde{z}''][b'+b''+v+1]![b+a''-v+1]!}$$
$$\times\Delta[\tfrac{1}{2}a',\tilde{I}'',\tfrac{1}{2}a]\sum_s \frac{q^{Q'_5+s((a'+a'')/2+\tilde{I}''+1)}}{[s]![\tilde{I}''+\tilde{z}''-s]![\tilde{I}''+\frac{1}{2}(a'-a)-s]!}$$
$$\times\frac{[b+a''-v+1+s]!}{[a-a'+v+s]![\frac{1}{2}(a'+a)-\tilde{I}''+a''+b-v+s+2]!} \tag{5.14}$$

(cf the expression with the triple sum derived by Asherova *et al* [49]). Here $\tilde{z}''=\frac{1}{2}(a'-a)-v$,

$$\Delta[a,b,c]=\left(\frac{[a+b-c]![a-b+c]![-a+b+c]!}{[a+b+c+1]!}\right)^{1/2} \tag{5.15}$$

$$Q'_5=-(b'+b''-b+v)^2-\tfrac{1}{2}\{(a'+a''+a-v)(a+b-2a''+2v)+bv-\tilde{I}''(\tilde{I}''+1)\}$$
$$+\tfrac{1}{8}(a-a')(a-a'+2)-(\tilde{I}''+\tilde{z}'')(\tilde{I}''-\tilde{z}''-a-b+v-1). \tag{5.16}$$

Now let us turn to the boundary canonical isofactors with an arbitrary value of t. We may use the recursive construction (cf Draayer and Akiyama [27], Ališauskas [10, 28]) of the independent twisted (Cornwell [50]) tensor operators

$$\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i''_z}=[T^{(kk)t,q}T^{(a''-k,b''-k)1,q}]^{(a''b'')q}_{y''i''i''_z} \tag{5.17}$$

derived by means of the stretched coupling of the self-adjoint canonical tensor operator $T^{(kk)t,q}_{y_2j_2m_2}$ of the minimal null space (with trivial shift of $\mathrm{u}_q(3)$ irreps and restricted shift of $\mathrm{u}_q(2)$ irreps for maximal $j_2=k=t-1$) and the tensor operator $T^{(a''-k,b''-k)1,q}_{y_1j_1m_1}$ with the maximal null space, ensuring the null space inclusion property of the $\mathrm{u}_q(3)$ canonical tensor operators (the vanishing of all their matrix elements for $t\leqslant\mathcal{M}-r$, where $\mathcal{M}$ is determined by (4.7)), after eliminating the superfluous tensor operators that appeared in (5.17) by means of the orthogonalization process begun from the maximal value of t.

Hence, by analogy with (2.13) and (5.1) of [28] the recoupling technique and our equations (5.7) and (4.13) allows us to derive the expansion coefficients of tensor operators $\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i''_z}$ as overlaps

$$\left(\tilde{T}^k|\eta_{-,+,\tilde{I}}\right)_q\equiv\sum_{t\geqslant k+1}U_3\begin{Bmatrix} (a'b') & \overset{1}{(a''-k,b''-k)} & (ab) \\ \overset{k+1}{(kk)} & (ab) & \overset{t}{(a''b'')} \end{Bmatrix}_q \begin{bmatrix} (a'b') & \overset{t}{(a''b'')} & (ab) \\ y'_0i'_0 & \bar{y}''_0\bar{i}''_0 & \tilde{y}\tilde{I} \end{bmatrix}_q^{(3)}$$
$$=([a+1][b+1][a+b+2][b']![a'+b'+1]![k]!)^{1/2}$$
$$\times\frac{[a''-k]![b''-k]!\Gamma[ab\tilde{I}\tilde{z}]\nabla[\bar{i}''_0i'_0\tilde{I}]}{\mathcal{D}\binom{q,t=1}{a''-k,b''-k}[a'b';ab]\mathcal{D}\binom{q,t=k+1}{kk}[ab;ab]\,([b'']!)^{1/2}}$$
$$\times\sum_{j,j'}\frac{(-1)^{(b''-a''-k)/2-\tilde{I}+j}q^{Q_6+j(j+1)-j'(j'+1)+3\tilde{I}(\tilde{I}+1)/2}}{\nabla^2[\frac{1}{2}(a''-k),\frac{1}{2}a',j]\nabla^2[\frac{1}{2}(a'+a''-a-k)-v,\frac{1}{2}a,j]}$$

$$\times \frac{[2j+1][2j'+1]}{[b-v+\frac{1}{2}(a'+a''-k)-j+1]![b-v+\frac{1}{2}(a'+a''-k)+j+2]!}$$
$$\times \frac{\nabla^2[\frac{1}{2}(a''+b'')-k,\, j',\, j]}{\nabla^2[\frac{1}{2}(b''-k),\, \frac{1}{2}a',\, j']\nabla^2[\frac{1}{2}k,\, j',\, \tilde{I}]\Gamma^2[abj'\tilde{z}-\frac{1}{2}k]} \tag{5.18}$$

where

$$Q_6 = Q_4(a', b', a''-k, b''-k, a, b) - \tfrac{1}{8}k^2 + \tfrac{3}{4}k + \tfrac{1}{2}k\tilde{z}. \tag{5.19}$$

The second sum (over j') now corresponds to the 'braided' q-factorial series, resembling the very well-poised ${}_{11}\phi_{10}$ basic hypergeometric series. A similar series also appears in the rather complicated expansion of overlaps $(\tilde{T}^k|\tilde{T}^{k'})_q$.

6. Concluding remarks

In this paper the roles of separate terminating classical and basic hypergeometric series and other q-factorial series in the boundary isoscalar factors and elementary overlaps of the coupling coefficients of SU(3) and $u_q(3)$ are demonstrated. The elementary overlap and its inverse in terms of the balanced basic ${}_5\phi_4$ or classical ${}_5F_4(1)$ hypergeometric series are derived (in section 3) for the first time for isofactors of both—$u_q(3)$ and SU(3), as well as the expansion of the extreme matrix elements of the (non-self-adjoint) canonical tensor operators with the minimal null space (in section 4). Otherwise the extension of some results [26, 28] from SU(3) to $u_q(3)$ and, especially, the q-phases, in sections 4 and 5 are rather non-trivial. The derivation of the corresponding equations (including their SU(3) versions) is considerably simplified, using special summation theorems of the very well-poised ${}_4\phi_3$ and ${}_6\phi_5$ [29], in addition to the summation theorem of ${}_2\phi_1$ (cf [39]). For a definite problem the Minton-type summation theorem [29] was usable, as well as the Karlsson-type summation theorem [29] which appeared when rearranging the multiplicity-free isofactors of $u_q(n)$ (see (3.15) of [8]). When the q-phases of overlaps (expressed in terms of the balanced and very well-poised basic hypergeometric series and leading to the paracanonical isofactors) do not depend on the summation parameters, the new class of q-factorial series, resembling the very well-poised basic hypergeometric series ${}_{2r+1}\phi_{2r}$ (but with the 'braiding' q-phase factor), appeared in the overlaps, associated with the canonical tensor operators. The used rearrangement and summation equations of these twisted q-factorial series, may probably be associated with some limit transitions for the very well-poised basic hypergeometric series ${}_{2r+2}\phi_{2r+1}$, by analogy with the limit transition between the Racah and Clebsch–Gordan coefficients of SU(2) or $u_q(2)$ and the Whipple [35] and Watson [36] transformation.

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